Sobolev-type inequalities on Riemannian manifolds with applications

Thesis for the Degree of Doctor of Philosophy

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Preface

Sobolev-type inequalities or more generally functional inequalities are often manifestations of natural physical phenomena as they often express very general laws of nature formulated in physics, biology, economics and engineering problems. They also form the basis of fundamental mathematical structures such as the calculus of variations. In order to study some elliptic problems one needs to exploit various Sobolev-type embeddings, to prove the lower semi-continuity of the energy functional or to prove that the energy functional satisfies the Palais-Smale condition. This is one of the reasons why calculus of variations is one of the most powerful and far-reaching tools available for advancing our understanding of mathematics and its applications.

The main objective of calculus of variations is the minimization of functionals, which has always been present in the real world in one form or another. I have carried out my research activity over the last years in the calculus of variations. More precisely we combined with my coauthors, elements from calculus of variations with PDE and with geometrical analysis to study some elliptic problems on curved spaces, with various nonlinearities (sub-linear, oscillatory etc.), see [23, 24, 25, 26, 27, 28, 37]. Such problems deserve as models for nonlinear phenomena coming from mathematical physics (solitary waves in Schrödinger or Schrödinger-Maxwell equations, etc.).

The main purpose of the present thesis is to present the recent achievements obtained in the theory of functional inequalities, more precisely to present some new Sobolev-type inequalities on Riemannian manifolds. More precisely, in the first part of the present thesis we focus on the theoretical part of the functional inequalities, while in the second part we present some applications of the theoretical achievements. Such developments are highly motivated from practical point of view supported by various examples coming from physics.

The thesis is based on the following papers:

- F. Faraci and C. Farkas. *New conditions for the existence of infinitely many solutions for a quasi-linear problem.* Proc. Edinb. Math. Soc. (2), 59(3):655–669, 2016.
- F. Faraci and C. Farkas. *A characterization related to Schrödinger equations on Riemannian manifolds.* ArXiv e-prints, April 2017.
- F. Faraci, C. Farkas, and A. Kristály. *Multipolar Hardy inequalities on Riemannian manifolds.* ESAIM Control Optim. Calc. Var., accepted, 2017, DOI: 10.1051/cocv/2017057.
- C. Farkas, *Schrödinger-Maxwell systems on compact Riemannian manifolds.* preprint, 2017.
- C. Farkas, J. Fodor, and A. Kristály. *Anisotropic elliptic problems involving sublinear terms.* In 2015 IEEE 10th Jubilee International Symposium on Applied Computational Intelligence and Informatics, pages 141–146, May 2015.
- C. Farkas and A. Kristály. *Schrödinger-Maxwell systems on non-compact Riemannian manifolds.* Nonlinear Anal. Real World Appl., 31:473–491, 2016.
- C. Farkas, A. Kristály, and A. Szakál. *Sobolev interpolation inequalities on Hadamard manifolds.* In Applied Computational Intelligence and Informatics (SACI), 2016 IEEE 11th International Symposium on, pages 161–165, May 2016.

Most of the results of the present thesis is stated for Cartan-Hadamard manifolds, despite the fact that they are valid for other geometrical structures as well. Although, any Cartan-Hadamard manifold (M, g) is diffeomorphic to $\mathbb{R}^n$, $n = \dim M$ (cf. Cartan's theorem), this is a wide class of non-compact Riemannian manifolds including important geometric objects (as Euclidean spaces, hyperbolic spaces, the space of symmetric positive definite matrices endowed with a suitable Killing metric), see Bridson and Haefliger [10].

We note that the structure of the present extended abstract is not the same as the structure of the PhD thesis. Therefore, for the sake of clarity, we sketch the structure of the PhD thesis.

In the first part of the thesis we present some theoretical achievements. We present here some surprising phenomena. In Chapter 1 we introduce the most important Sobolev inequalities both on the Euclidean and on Riemannian settings.

In Chapter 2 we prove Sobolev-type interpolation inequalities on Cartan-Hadamard manifolds and their optimality whenever the Cartan-Hadamard conjecture holds (e.g., in dimensions 2, 3 and 4). The existence of extremals leads to unexpected rigidity phenomena. This chapter is based on the paper [29].

In Chapter 3 we prove some multipolar Hardy inequalities on complete Riemannian manifolds, providing various curved counterparts of some Euclidean multipolar inequalities due to Cazacu and Zuazua [13]. We notice that our inequalities deeply depend on the curvature, providing (quantitative) information about the deflection from the flat case. This chapter is based on the recent paper [24].

In the second part of the thesis we present some applications, namely we study some PDE's on Riemannian manifolds. In Chapter 5 we study nonlinear Schrödinger-Maxwell systems on 3-dimensional compact Riemannian manifolds proving a new kind of multiplicity result with sublinear and superlinear nonlinearities. This chapter is based on [25].

In Chapter 6, we consider a Schrödinger-Maxwell system on n-dimensional Cartan-Hadamard manifolds, where $3 \leq n \leq 5$. The main difficulty resides in the lack of compactness of such manifolds which is recovered by exploring suitable isometric actions. By combining variational arguments, some existence, uniqueness and multiplicity of isometry-invariant weak solutions are established for such systems depending on the behavior of the nonlinear term. We also present a new set of assumptions ensuring the existence of infinitely many solutions for a quasilinear equation, which can be adapted easily to Schrödinger-Maxwell systems. This Chapter is based on the papers [22, 26].

In Chapter 7, by using inequalities presented in Chapter 2, together with variational methods, we also establish non-existence, existence and multiplicity results for certain Schrödinger-type problems involving the Laplace-Beltrami operator and bipolar potentials on Cartan-Hadamard manifolds. We also mention a multiplicity result for an anisotropic sub-linear elliptic problem with Dirichlet boundary condition, depending on a positive parameter λ. We prove that for enough large values of λ, our anisotropic problem has at least two non-zero distinct solutions. In particular, we show that at least one of the solutions provides a Wulff-type symmetry. This Chapter is based on the papers [24, 27].

In Chapter 8, we consider a Schrödinger type equation on non-compact Riemannian manifolds, depending on a positive parameter λ. By using variational methods we prove a characterization result for existence of solutions for this problem. This chapter is based on the paper [23].

1

Sobolev interpolation inequalities on Cartan-Hadamard manifolds

> The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful.
>
> *(Henri Poincaré)*

1.1 Statement of main results

Let $n \geq 2$, $p \in (1, n)$, $1 < \alpha \leq \frac{n}{n-p}$ and $\theta = \frac{p^\star(\alpha-1)}{\alpha p(p^\star - \alpha p + \alpha - 1)}$. Then the optimal Gagliardo-Nirenberg interpolation inequality states that

$$\|u\|_{L^{\alpha p}} \leq \mathcal{G}_{\alpha,p,n} \|\nabla u\|_{L^p}^{\theta} \|u\|_{L^{\alpha(p-1)+1}}^{1-\theta}, \ \forall u \in C_0^\infty(\mathbb{R}^n),$$

where the optimal constant $\mathcal{G}_{\alpha,p,n}$ is given by

$$\mathcal{G}_{\alpha,p,n} = \left(\frac{\alpha-1}{p'}\right)^{\theta} \frac{\left(\frac{p'}{n}\right)^{\frac{\theta}{p}+\frac{\theta}{n}} \left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}\right)^{\frac{1}{\alpha p}} \left(\frac{\alpha(p-1)+1}{\alpha-1}\right)^{\frac{\theta}{p}-\frac{1}{\alpha p}}}{\left(\omega_n \mathsf{B}\left(\frac{\alpha(p-1)+1}{\alpha-1} - \frac{n}{p'}, \frac{n}{p'}\right)\right)^{\frac{\theta}{n}}},$$

B is the Euler beta-function and ω_n is the volume of the $n-$dimensional Euclidean unit ball. The previous inequality reduces to the optimal Sobolev inequality when $\alpha = \frac{n}{n-p}$, see Talenti [45] and Aubin [2]. We also note that the families of extremal functions are *uniquely* determined up to translation, constant multiplication and scaling, see Cordero-Erausquin, Nazaret and Villani [15], Del Pino and Dolbeault [20].

Recently, Kristály [38] studied Gagliardo-Nirenberg inequalities on a generic metric measure space which satisfies the Lott-Sturm-Villani curvature-dimension condition $\mathsf{CD}(K, n)$ for some $K \geq 0$ and $n \geq 2$, by establishing some global non-collapsing $n-$dimensional volume growth properties.

The purpose of the present chapter is study the counterpart of the aforementioned paper; namely, we shall consider spaces which are non-positively curved.

To be more precise, let (M, g) be an $n(\geq 2)-$dimensional Cartand-Hadamard manifold (i.e., a complete, simply connected Riemannian manifold with non-positive sectional curvature) endowed with its canonical volume form dv_g. We say that the *Cartan-Hadamard conjecture holds on* (M, g) if

$$\text{Area}_g(\partial D) \geq n\omega_n^{\frac{1}{n}} \text{Vol}_g(D)^{\frac{n-1}{n}} \tag{1.1.1}$$

for any bounded domain $D \subset M$ with smooth boundary ∂D and equality holds in (1.1.1) if and only if D is isometric to the $n-$dimensional Euclidean ball with volume $\text{Vol}_g(D)$, see Aubin [2]. Note that $n\omega_n^{\frac{1}{n}}$ is precisely the isoperimetric ratio in the Euclidean setting. Hereafter, $\text{Area}_g(\partial D)$ stands for the area of ∂D with respect to the metric induced on ∂D by g, and

$\text{Vol}_g(D)$ is the volume of D with respect to g. We note that the Cartan-Hadamard conjecture is true in dimension 2 (cf. Beckenbach and Radó [7] in dimension 3 (cf. Kleiner [33]); and in dimension 4 (cf. Croke [16]), but it is open for higher dimensions.

For $n \geq 3$, Croke [16] proved a general isoperimetric inequality on Hadamard manifolds:

$$\text{Area}_g(\partial D) \geq C(n)\text{Vol}_g(D)^{\frac{n-1}{n}} \tag{1.1.2}$$

for any bounded domain $D \subset M$ with smooth boundary ∂D, where

$$C(n) = (n\omega_n)^{1-\frac{1}{n}} \left((n-1)\omega_{n-1} \int_0^{\frac{\pi}{2}} \cos^{\frac{n}{n-2}}(t) \sin^{n-2}(t) dt \right)^{\frac{2}{n}-1}. \tag{1.1.3}$$

Note that $C(n) \leq n\omega_n^{\frac{1}{n}}$ for every $n \geq 3$ while equality holds if and only if $n = 4$. Let $C(2) = 2\sqrt{\pi}$.

Our main results can be stated as follows:

Theorem 1.1.1 (Farkas, Kristály and Szakál [29]). *Let (M, g) be an $n(\geq 2)-$dimensional Cartan-Hadamard manifold, $p \in (1, n)$ and $\alpha \in (1, \frac{n}{n-p}]$. Then we have:*

(i) *The Gagliardo-Nirenberg inequality*

$$\|u\|_{L^{\alpha p}(M)} \leq \mathcal{C}\|\nabla_g u\|^{\theta}_{L^p(M)} \|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)}, \ \forall u \in C_0^\infty(M) \tag{$\mathbf{GN1}$}_{\mathcal{C}}^{\alpha,p}$$

holds for $\mathcal{C} = \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)} \right)^{\theta} \mathcal{G}_{\alpha,p,n}$;

(ii) *If the Cartan-Hadamard conjecture holds on (M, g), then the optimal Gagliardo-Nirenberg inequality* $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ *holds on (M, g), i.e.,*

$$\mathcal{G}^{-1}_{\alpha,p,n} = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\|\nabla_g u\|^{\theta}_{L^p(M)} \|u\|^{1-\theta}_{L^{\alpha(p-1)+1}(M)}}{\|u\|_{L^{\alpha p}(M)}}. \tag{1.1.4}$$

In almost similar way, we can prove the following result:

Theorem 1.1.2 (Farkas, Kristály and Szakál [29]). *Let (M, g) be an $n(\geq 2)-$dimensional Cartan-Hadamard manifold, $p \in (1, n)$ and $\alpha \in (0, 1)$. Then we have:*

(i) *The Gagliardo-Nirenberg inequality*

$$\|u\|_{L^{\alpha(p-1)+1}(M)} \leq \mathcal{C}\|\nabla_g u\|^{\gamma}_{L^p(M)} \|u\|^{1-\gamma}_{L^{\alpha p}(M)}, \ \forall u \in \text{Lip}_0(M) \tag{$\mathbf{GN2}$}_{\mathcal{C}}^{\alpha,p}$$

holds for $\mathcal{C} = \left(\frac{n\omega_n^{\frac{1}{n}}}{C(n)} \right)^{\gamma} \mathcal{N}_{\alpha,p,n}$;

(ii) *If the Cartan-Hadamard conjecture holds on (M, g), then the optimal Gagliardo-Nirenberg inequality* $(\mathbf{GN2})^{\alpha,p}_{\mathcal{N}_{\alpha,p,n}}$ *holds on (M, g), i.e.,*

$$\mathcal{N}^{-1}_{\alpha,p,n} = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\|\nabla_g u\|^{\gamma}_{L^p(M)} \|u\|^{1-\gamma}_{L^{\alpha p}(M)}}{\|u\|_{L^{\alpha(p-1)+1}(M)}}.$$

Before to state the last result of this section, we need one more notion (see Kristály [36]): a function $u : M \to [0, \infty)$ is *concentrated around* $x_0 \in M$ if for every $0 < t < \|u\|_{L^\infty}$ the level set $\{x \in M : u(x) > t\}$ is a geodesic ball $B_{x_0}(r_t)$ for some $r_t > 0$. Note that in $\mathbb{R}^n$ (see [15]) the extremal function is concentrated around the origin. Now we are in the position to state the following characterization concerning the extremals:

Theorem 1.1.3 (Farkas, Kristály and Szakál [29]). *Let (M, g) be an $n(\geq 2)-$dimensional Cartan-Hadamard manifold which satisfies the Cartan-Hadamard conjecture, $p \in (1, n)$ and $x_0 \in M$. The following statements are equivalent:*

(i) *For a fixed $\alpha \in \left(1, \dfrac{n}{n-p}\right]$, there exists a bounded positive extremal function in $(\mathbf{GN1})^{\alpha,p}_{\mathcal{G}_{\alpha,p,n}}$ concentrated around x_0;*

(ii) *For a fixed $\alpha \in \left(\dfrac{1}{p}, 1\right)$, to every $\lambda > 0$ there exists a non-negative extremal function $u_\lambda \in C_0^\infty(M)$ in $(\mathbf{GN2})^{\alpha,p}_{\mathcal{N}_{\alpha,p,n}}$ concentrated around x_0 and $\mathrm{Vol}_g(\mathrm{supp}(u_\lambda)) = \lambda$;*

(iii) *(M, g) is isometric to the Euclidean space $\mathbb{R}^n$.*

2

Multipolar Hardy inequalities on Riemannian manifolds

True pleasure lies not in the discovery of truth, but in the search for it.

(Tolstoy)

2.1 Introduction and statement of main results

The classical *unipolar Hardy inequality* (or, uncertainty principle) states that if $n \geq 3$, then

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \mathrm{d}x, \ \forall u \in C_0^\infty(\mathbb{R}^n);$$

here, the constant $\frac{(n-2)^2}{4}$ is sharp and not achieved. Many efforts have been made over the last two decades to improve/extend Hardy inequalities in various directions. One of the most challenging research topics in this direction is the so-called *multipolar Hardy inequality*. Such kind of extension is motivated by molecular physics and quantum chemistry/cosmology. Indeed, by describing the behavior of electrons and atomic nuclei in a molecule within the theory of Born-Oppenheimer approximation or Thomas-Fermi theory, particles can be modeled as certain singularities/poles $x_1, ..., x_m \in \mathbb{R}^n$, producing their effect within the form $x \mapsto |x - x_i|^{-1}$, $i \in \{1, ..., m\}$.

Recently, Cazacu and Zuazua [13] proved an optimal multipolar counterpart of the above (unipolar) Hardy inequality, i.e.,

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \frac{|x_i - x_j|^2}{|x - x_i|^2 |x - x_j|^2} u^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{2.1.1}$$

where $n \geq 3$, and $x_1, ..., x_m \in \mathbb{R}^n$ are different poles; moreover, the constant $\frac{(n-2)^2}{m^2}$ is optimal. By using the paralelogrammoid law, (2.1.1) turns to be equivalent to

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i < j \leq m} \int_{\mathbb{R}^n} \left| \frac{x - x_i}{|x - x_i|^2} - \frac{x - x_j}{|x - x_j|^2} \right|^2 u^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n). \tag{2.1.2}$$

In the sequel we shall present our results; for further use, let Δ_g be the Laplace-Beltrami operator on (M, g). Let $m \geq 2$, $S = \{x_1, ..., x_m\} \subset M$ be the set of poles with $x_i \neq x_j$ if $i \neq j$, and for simplicity of notation, let $d_i = d_g(\cdot, x_i)$ for every $i \in \{1, ..., m\}$. Our main result reads as follows.

Theorem 2.1.1 (Faraci, Farkas and Kristály [24]). *Let (M, g) be an n-dimensional complete Riemannian manifold and $S = \{x_1, ..., x_m\} \subset M$ be the set of distinct poles, where $n \geq 3$ and $m \geq 2$. Then*

$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \mathrm{d}v_g + \frac{n-2}{m} \sum_{i=1}^{m} \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \mathrm{d}v_g, \quad \forall u \in C_0^\infty(M). \tag{2.1.3}$$

Moreover, in the bipolar case (i.e., $m = 2$), the constant $\frac{(n-2)^2}{m^2} = \frac{(n-2)^2}{4}$ is optimal in (2.1.3).

For further use, we notice that $\mathbf{K} \geq c$ (resp. $\mathbf{K} \leq c$) means that the sectional curvature on (M, g) is bounded from below (resp. above) by $c \in \mathbb{R}$ at any point and direction.

2.2 A bipolar Schrödinger-type equation on Cartan-Hadamard manifolds

Using inequality (2.1.3), we obtain the following non-positively curved versions of Cazacu and Zuazua's inequalities (2.1.2) and (2.1.1) for multiple poles, respectively:

Corollary 2.2.1 (Faraci, Farkas and Kristály [24]). *Let (M, g) be an n-dimensional Cartan-Hadamard manifold and let $S = \{x_1, ..., x_m\} \subset M$ be the set of distinct poles, with $n \geq 3$ and $m \geq 2$. Then we have the following inequality:*

$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|^2 u^2 \mathrm{d}v_g, \quad \forall u \in H_g^1(M). \tag{2.2.1}$$

Moreover, if $\mathbf{K} \geq k_0$ for some $k_0 \in \mathbb{R}$, then

$$\int_M |\nabla_g u|^2 \mathrm{d}v_g \geq \frac{4(n-2)^2}{m^2} \sum_{1\leq i<j\leq m} \int_M \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{ij}}{2}\right)}{d_i d_j \mathbf{s}_{k_0}(d_i) \mathbf{s}_{k_0}(d_j)} u^2 \mathrm{d}v_g, \quad \forall u \in H_g^1(M). \tag{2.2.2}$$

2.3 Singular Schrödinger type equations on Cartan-Hadamard manifolds

In this section we present an application of the inequalities presented above.

In the sequel, let (M, g) be an n-dimensional Cartan-Hadamard manifold ($n \geq 3$) with $\mathbf{K} \geq k_0$ for some $k_0 \leq 0$, and $S = \{x_1, x_2\} \subset M$ be the set of poles. In this section we deal with the Schrödinger-type equation

$$-\Delta_g u + V(x)u = \lambda \frac{\mathbf{s}_{k_0}^2\left(\frac{d_{12}}{2}\right)}{d_1 d_2 \mathbf{s}_{k_0}(d_1)\mathbf{s}_{k_0}(d_2)} u + \mu W(x) f(u) \quad \text{in } M, \tag{$\mathscr{P}_M^\mu$}$$

where $\lambda \in \left[0, (n-2)^2\right)$ is fixed, $\mu \geq 0$ is a parameter, and the continuous function $f : [0, \infty) \to \mathbb{R}$ verifies

(f_1) $f(s) = o(s)$ as $s \to 0^+$ and $s \to \infty$;

(f_2) $F(s_0) > 0$ for some $s_0 > 0$, where $F(s) = \int_0^s f(t)\mathrm{d}t$.

According to (f_1) and (f_2), the number $c_f = \max_{s>0} \frac{f(s)}{s}$ is well defined and positive.

On the potential $V : M \to \mathbb{R}$ we require that

(V_1) $V_0 = \inf_{x \in M} V(x) > 0$;

(V_2) $\lim_{d_g(x_0,x) \to \infty} V(x) = +\infty$ for some $x_0 \in M$,

and $W : M \to \mathbb{R}$ is assumed to be positive.

Before to state our result, let us consider the functional space

$$H^1_V(M) = \left\{ u \in H^1_g(M) : \int_M \left(|\nabla_g u|^2 + V(x)u^2 \right) \mathrm{d}v_g < +\infty \right\}$$

endowed with the norm

$$\|u\|_V = \left(\int_M |\nabla_g u|^2 \, \mathrm{d}v_g + \int_M V(x)u^2 \, \mathrm{d}v_g \right)^{1/2}.$$

The main result of this subsection is as follows.

Theorem 2.3.1 (Faraci, Farkas and Kristály [24]). *Let (M, g) be an n-dimensional Cartan-Hadamard manifold ($n \geq 3$) with $\mathbf{K} \geq k_0$ for some $k_0 \leq 0$ and let $S = \{x_1, x_2\} \subset M$ be the set of distinct poles. Let $V, W : M \to \mathbb{R}$ be positive potentials verifying (V_1), (V_2) and $W \in L^1(M) \cap L^\infty(M) \setminus \{0\}$, respectively. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function verifying (f_1) and (f_2), and $\lambda \in \left[0, (n-2)^2\right)$ be fixed. Then the following statements hold:*

(i) *Problem $(\mathscr{P}^\mu_M)$ has only the zero solution whenever $0 \leq \mu < V_0 \|W\|^{-1}_{L^\infty(M)} c_f^{-1}$;*

(ii) *There exists $\mu_0 > 0$ such that problem $(\mathscr{P}^\mu_M)$ has at least two distinct non-zero, non-negative weak solutions in $H^1_V(M)$ whenever $\mu > \mu_0$.*

3

Schrödinger-Maxwell systems

Whatever you do may seem insignificant to you, but it is most important that you do it.

(Gandhi)

3.1 Introduction and motivation

The Schrödinger-Maxwell system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + eu\phi = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = 4\pi e u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.1.1}$$

describes the statical behavior of a charged non-relativistic quantum mechanical particle interacting with the electromagnetic field. More precisely, the unknown terms $u : \mathbb{R}^3 \to \mathbb{R}$ and $\phi : \mathbb{R}^3 \to \mathbb{R}$ are the fields associated to the particle and the electric potential, respectively. Here and in the sequel, the quantities m, e, ω and $\hbar$ are the mass, charge, phase, and Planck's constant, respectively, while $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function verifying some growth conditions.

In fact, system (3.1.1) comes from the evolutionary nonlinear Schrödinger equation by using a Lyapunov-Schmidt reduction.

The Schrödinger-Maxwell system (or its variants) has been the object of various investigations in the last two decades. Without sake of completeness, we recall in the sequel some important contributions to the study of system (3.1.1). Benci and Fortunato [9] considered the case of $f(x,s) = |s|^{p-2}s$ with $p \in (4,6)$ by proving the existence of infinitely many radial solutions for (3.1.1); their main step relies on the reduction of system (3.1.1) to the investigation of critical points of a "one-variable" energy functional associated with (3.1.1). Based on the idea of Benci and Fortunato, under various growth assumptions on f further existence/multiplicity results can be found in Ambrosetti and Ruiz [1], Azzolini [3], Azzollini, d'Avenia and Pomponio [4], d'Avenia [19], d'Aprile and Mugnai [17], Cerami and Vaira [14], Kristály and Repovs [39], Ruiz [43] and references therein. By means of a Pohozaev-type identity, d'Aprile and Mugnai [18] proved the non-existence of non-trivial solutions to system (3.1.1) whenever $f \equiv 0$ or $f(x,s) = |s|^{p-2}s$ and $p \in (0,2] \cup [6,\infty)$.

In the last five years Schrödinger-Maxwell systems has been studied on $n-$dimensional *compact Riemannian manifolds* $(2 \le n \le 5)$ by Druet and Hebey [21], Hebey and Wei [32], Ghimenti and Micheletti [30, 31] and Thizy [46, 47]. More precisely, in the aforementioned papers various forms of the system

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + \omega u + eu\phi = f(u) & \text{in } M, \\ -\Delta_g\phi + \phi = 4\pi e u^2 & \text{in } M, \end{cases} \tag{3.1.2}$$

has been considered, where (M,g) is a compact Riemannian manifold and Δ_g is the Laplace-Beltrami operator, by proving existence results with further qualitative property of the solution(s). As expected, the compactness of (M,g) played a crucial role in these investigations.

3.2 Schrödinger-Maxwell systems: the compact case

In this section we are focusing to the following Schrödinger-Maxwell system:

$$\begin{cases} -\Delta_g u + \beta(x)u + eu\phi = \Psi(\lambda, x)f(u) & \text{in} \quad M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in} \quad M, \end{cases} \tag{$\mathcal{SM}^e_{\Psi(\lambda,\cdot)}$}$$

where (M, g) is 3-dimensional compact Riemannian manifold without boundary, $e, q > 0$ are positive numbers, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\beta \in C^\infty(M)$ and $\Psi \in C^\infty(\mathbb{R}_+ \times M)$ are positive functions. The solutions (u, ϕ) of $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ are sought in the Sobolev space $H^1_g(M) \times H^1_g(M)$.

We first consider a continuous function $f : [0, \infty) \to \mathbb{R}$ which verifies the following assumptions:

(f_1) $\frac{f(s)}{s} \to 0$ as $s \to 0^+$;

(f_2) $\frac{f(s)}{s} \to 0$ as $s \to \infty$;

(f_3) $F(s_0) > 0$ for some $s_0 > 0$, where $F(s) = \displaystyle\int_0^s f(t)\mathrm{d}t$, $s \geq 0$.

Due to the assumptions $(f_1) - (f_3)$, the numbers

$$c_f = \max_{s>0} \frac{f(s)}{s}$$

and

$$c_F = \max_{s>0} \frac{4F(s)}{2s^2 + eqs^4}$$

are well-defined and positive. Now, we are in the position to state the first result of the paper.

Theorem 3.2.1 (Farkas [25]). *Let (M, g) be $3-$dimensional compact Riemannian manifold without boundary, and let $\beta \equiv 1$. Assume that $\Psi(\lambda, x) = \lambda\alpha(x)$ and $\alpha \in L^\infty(M)$ is a positive function. If the continuous function $f : [0, \infty) \to \mathbb{R}$ satisfies assumptions $(f_1) - (f_3)$, then*

(a) *if $0 \leq \lambda < c_f^{-1} \|\alpha\|^{-1}_{L^\infty}$, system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has only the trivial solution;*

(b) *for every $\lambda \geq c_F^{-1} \|\alpha\|^{-1}_{L^1}$, system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has at least two distinct non-zero, non-negative weak solutions in $H^1_g(M) \times H^1_g(M)$.*

In order to obtain new kind of multiplicity result for the system $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ instead of the assumption (f_1) we require the following one:

(f_4) There exists $\mu_0 > 0$ such that the set of all global minima of the function

$$t \mapsto \Phi_{\mu_0}(t) := \frac{1}{2}t^2 - \mu_0 F(t)$$

has at least $m \geq 2$ connected components.

In this case we can state the following result:

Theorem 3.2.2 (Farkas [25]). *Let (M, g) be an $3-$dimensional compact Riemannian manifold without boundary. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function which satisfies (f_2) and (f_4), $\beta \in C^\infty(M)$ is a positive function. Assume that $\Psi(\lambda, x) = \lambda\alpha(x) + \mu_0\beta(x)$, where $\alpha \in C^\infty(M)$ is a positive function. Then for every $\tau > \max\{0, \|\alpha\|_{L^1(M)} \max_t \Phi_{\mu_0}(t)\}$ there exists $\lambda_\tau > 0$ such that for every $\lambda \in (0, \lambda_\tau)$ the problem $(\mathcal{SM}^\lambda_{\Psi(\lambda,\cdot)})$ has at least $m + 1$ solutions.*

As a counterpart of the Theorem 3.2.1 we consider the case when the continuous function $f : [0, +\infty) \to \mathbb{R}$ satisfies the following assumptions:

($\tilde{f}_1$) $|f(s)| \leq C(s + s^{p-1})$, for all $s \in [0, +\infty)$, where $C > 0$ and $p \in (4, 6)$;

($\tilde{f}_2$) there exists $\eta > 4$ and $\tau_0 > 0$ such that

$$0 < \eta F(s) \leq s f(s), \forall s \geq \tau_0.$$

Theorem 3.2.3 (Farkas [25]). *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function, which satisfies hypotheses ($\tilde{f}_1$), ($\tilde{f}_2$). Then there exists λ_0 such that for every $0 < \lambda < \lambda_0$ the problem $(\mathcal{SM}^e_{\Psi(\lambda,\cdot)})$ has at least two solutions.*

3.3 Schrödinger-Maxwell systems: the non-compact case

We shall consider the Schrödinger-Maxwell system

$$\begin{cases} -\Delta_g u + u + eu\phi = \lambda\alpha(x)f(u) & \text{in} \quad M, \\ -\Delta_g \phi + \phi = qu^2 & \text{in} \quad M, \end{cases} \tag{$\mathcal{SM}_\lambda$}$$

where (M, g) is an $n-$dimensional Cartan-Hadamard manifold ($3 \leq n \leq 5$), $e, q > 0$ are positive numbers, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\alpha : M \to \mathbb{R}$ is a measurable function, and $\lambda > 0$ is a parameter. The solutions (u, ϕ) of $(\mathcal{SM}_\lambda)$ are sought in the Sobolev space $H^1_g(M) \times H^1_g(M)$. In the sequel, we shall formulate rigourously our main results with some comments.

The pair $(u, \phi) \in H^1_g(M) \times H^1_g(M)$ is a *weak solution* to the system $(\mathcal{SM}_\lambda)$ if

$$\int_M (\langle \nabla_g u, \nabla_g v\rangle + uv + eu\phi v)\mathrm{d}v_g = \lambda \int_M \alpha(x)f(u)v\mathrm{d}v_g \text{ for all } v \in H^1_g(M), \tag{3.3.1}$$

$$\int_M (\langle \nabla_g \phi, \nabla_g \psi\rangle + \phi\psi)\mathrm{d}v_g = q \int_M u^2\psi\mathrm{d}v_g \text{ for all } \psi \in H^1_g(M). \tag{3.3.2}$$

For later use, we denote by $\mathrm{Isom}_g(M)$ the group of isometries of (M, g) and let G be a subgroup of $\mathrm{Isom}_g(M)$. A function $u : M \to \mathbb{R}$ is *$G-$invariant* if $u(\sigma(x)) = u(x)$ for every $x \in M$ and $\sigma \in G$. Furthermore, $u : M \to \mathbb{R}$ is *radially symmetric w.r.t.* $x_0 \in M$ if u depends on $d_g(x_0, \cdot)$, d_g being the Riemannian distance function. The *fixed point set of G on M* is given by $\mathrm{Fix}_M(G) = \{x \in M : \sigma(x) = x \text{ for all } \sigma \in G\}$. For a given $x_0 \in M$, we introduce the following hypothesis which will be crucial in our investigations:

$(\boldsymbol{H^{x_0}_G})$ *The group G is a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G) = \{x_0\}$.*

For $x_0 \in M$ fixed, we also introduce the hypothesis

$(\boldsymbol{\alpha^{x_0}})$ *The function $\alpha : M \to \mathbb{R}$ is non-zero, non-negative and radially symmetric w.r.t. x_0.*

Our results are divided into two classes:

A. Schrödinger-Maxwell systems of Poisson type. Dealing with a Poisson-type system, we set $\lambda = 1$ and $f \equiv 1$ in $(\mathcal{SM}_\lambda)$. For abbreviation, we simply denote $(\mathcal{SM}_1)$ by $(\mathcal{SM})$.

Theorem 3.3.1 (Farkas and Kristály [26]). *Let (M, g) be an $n-$dimensional homogeneous Cartan-Hadamard manifold ($3 \leq n \leq 6$), and $\alpha \in L^2(M)$ be a non-negative function. Then there exists a unique, non-negative weak solution $(u_0, \phi_0) \in H^1_g(M) \times H^1_g(M)$ to problem $(\mathcal{SM})$. Moreover, if $x_0 \in M$ is fixed and α satisfies $(\boldsymbol{\alpha^{x_0}})$, then (u_0, ϕ_0) is $G-$invariant w.r.t. any group $G \subset \mathrm{Isom}_g(M)$ which satisfies $(\boldsymbol{H^{x_0}_G})$.*

For $c \leq 0$ and $3 \leq n \leq 6$ we consider the ordinary differential equations system

$$\begin{cases} -h_1''(r) - (n-1)\mathbf{ct_c}(s)h_1'(r) + h_1(r) + eh_1(r)h_2(r) - \alpha_0(r) = 0, \ r \geq 0; \\ -h_2''(r) - (n-1)\mathbf{ct_c}(r)h_2'(r) + h_2(r) - qh_1(r)^2 = 0, \ r \geq 0; \\ \displaystyle\int_0^\infty (h_1'(r)^2 + h_1^2(r))\mathbf{s}_c(r)^{n-1}\mathrm{d}r < \infty; \\ \displaystyle\int_0^\infty (h_2'(r)^2 + h_2^2(r))\mathbf{s}_c(r)^{n-1}\mathrm{d}r < \infty, \end{cases} \tag{$\mathscr{R}$}$$

where $\alpha_0 : [0,\infty) \to [0,\infty)$ satisfies the integrability condition $\alpha_0 \in L^2([0,\infty), \mathbf{s}_c(r)^{n-1}\mathrm{d}r)$.

The system $(\mathscr{R})$ has a unique, non-negative solution $(h_1^c, h_2^c) \in C^\infty(0,\infty) \times C^\infty(0,\infty)$. In fact, the following rigidity result can be stated:

Theorem 3.3.2 (Farkas and Kristály [26]). *Let (M,g) be an $n-$dimensional homogeneous Cartan-Hadamard manifold $(3 \leq n \leq 6)$ with sectional curvature $\mathbf{K} \leq c \leq 0$. Let $x_0 \in M$ be fixed, and $G \subset \mathrm{Isom}_g(M)$ and $\alpha \in L^2(M)$ be such that hypotheses $(\boldsymbol{H_G^{x_0}})$ and $(\boldsymbol{\alpha^{x_0}})$ are satisfied. If $\alpha^{-1}(t) \subset M$ has null Riemannian measure for every $t \geq 0$, then the following statements are equivalent:*

(i) $(h_1^c(d_g(x_0,\cdot)), h_2^c(d_g(x_0,\cdot)))$ *is the unique pointwise solution of* $(\mathcal{SM})$;

(ii) (M,g) *is isometric to the space form with constant sectional curvature* $\mathbf{K} = c$.

B. Schrödinger-Maxwell systems involving oscillatory terms. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function with $F(s) = \displaystyle\int_0^s f(t)\mathrm{d}t$. We assume:

(f_0^1) $-\infty < \liminf\limits_{s\to 0} \dfrac{F(s)}{s^2} \leq \limsup\limits_{s\to 0} \dfrac{F(s)}{s^2} = +\infty$;

(f_0^2) there exists a sequence $\{s_j\}_j \subset (0,1)$ converging to 0 such that $f(s_j) < 0$, $j \in \mathbb{N}$.

Theorem 3.3.3 (Farkas and Kristály [26]). *Let (M,g) be an $n-$dimensional homogeneous Cartan-Hadamard manifold $(3 \leq n \leq 5)$, $x_0 \in M$ be fixed, and $G \subset \mathrm{Isom}_g(M)$ and $\alpha \in L^1(M) \cap L^\infty(M)$ be such that hypotheses $(\boldsymbol{H_G^{x_0}})$ and $(\boldsymbol{\alpha^{x_0}})$ are satisfied. If $f : [0,\infty) \to \mathbb{R}$ is a continuous function satisfying (f_0^1) and (f_0^2), then there exists a sequence $\{(u_j^0, \phi_{u_j^0})\}_j \subset H_g^1(M) \times H_g^1(M)$ of distinct, non-negative $G-$invariant weak solutions to $(\mathcal{SM})$ such that*

$$\lim_{j\to\infty} \|u_j^0\|_{H_g^1(M)} = \lim_{j\to\infty} \|\phi_{u_j^0}\|_{H_g^1(M)} = 0.$$

4

A characterization related to Schrödinger equations on Riemannian manifolds

> I hear and I forget. I see and I remember. I do and I understand.
>
> *(Confucius)*

4.1 Introduction and statement of main results

The existence of standing waves solutions for the nonlinear Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - f(x, |\psi|), \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \setminus \{0\},$$

has been intensively studied in the last decades. The Schrödinger equation plays a central role in quantum mechanic as it predicts the future behavior of a dynamic system. Indeed, the wave function $\psi(x,t)$ represents the quantum mechanical probability amplitude for a given unit-mass particle to have position x at time t. Such equation appears in several fields of physics, from Bose–Einstein condensates and nonlinear optics, to plasma physics (see for instance Byeon and Wang [11] and Cao, Noussair and Yan [12] and reference therein).

A Lyapunov-Schmidt type reduction, i.e. a separation of variables of the type $\psi(x,t) = u(x)e^{-i\frac{E}{\hbar}t}$, leads to the following semilinear elliptic equation

$$-\Delta u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^n.$$

With the aid of variational methods, the existence and multiplicity of nontrivial solutions for such problems have been extensively studied in the literature over the last decades. For instance, the existence of positive solutions when the potential V is coercive and f satisfies standard mountain pass assumptions, are well known after the seminal paper of Rabinowitz [42]. Moreover, in the class of bounded from below potentials, several attempts have been made to find general assumptions on V in order to obtain existence and multiplicity results (see for instance Bartsch, Pankov and Wang [6], Bartsch and Wang [5], Benci and Fortunato [8] Willem [48] and Strauss [44]). In such papers the nonlinearity f is required to satisfy the well-know Ambrosetti-Rabinowitz condition, thus it is superlinear at infinity. For a sublinear growth of f see also Kristály [34].

Most of the aforementioned papers provide *sufficient* conditions on the nonlinear term f in order to prove existence/multiplicity type results. The novelty of the present chapter is to establish a *characterization* result for stationary Schrödinger equations on unbounded domains; even more, our arguments work on not necessarily linear structures. Indeed, our results fit the research direction where the solutions of certain PDEs are influenced by the geometry of the ambient structure (see for instance Farkas, Kirstály and Varga [28], Farkas and Kristály [26], Kristály [35], Li and Yau [40], Ma [41] and reference therein). Accordingly, we deal with a Riemannian setting, the results on $\mathbb{R}^n$ being a particular consequence of our general achievements.

Let $x_0 \in M$ be a fixed point, $\alpha : M \to \mathbb{R}_+ \setminus \{0\}$ a bounded function and $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$ such that there exist two constants $C > 0$ and $q \in (1, 2^\star)$ (being $2^\star$ the Sobolev critical exponent) such that

$$f(\xi) \leq k\left(1 + \xi^{q-1}\right) \text{ for all } \xi \geq 0. \tag{4.1.1}$$

Denote by $F : \mathbb{R}_+ \to \mathbb{R}_+$ the function $F(\xi) = \displaystyle\int_0^\xi f(t)dt$.

We assume that $V : M \to \mathbb{R}$ is a measurable function satisfying the following conditions:

(V_1) $V_0 = \operatorname{essinf}_{x \in M} V(x) > 0$;

(V_2) $\displaystyle\lim_{d_g(x_0,x)\to\infty} V(x) = +\infty$, for some $x_0 \in M$.

The problem we deal with is written as:

$$\begin{cases} -\Delta_g u + V(x)u = \lambda\alpha(x)f(u), & \text{in } M \\ u \geq 0, & \text{in } M \\ u \to 0, & \text{as } d_g(x_0, x) \to \infty. \end{cases} \tag{$\mathscr{P}_\lambda$}$$

Our result reads as follows:

Theorem 4.1.1 (Faraci and Farkas [23]). *Let $n \geq 3$ and (M, g) be a complete, non-compact $n-$dimensional Riemannian manifold satisfying the curvature condition* $(\mathbf{C})$, *and* $\inf_{x\in M} \operatorname{Vol}_g(B_x(1)) > 0$. *Let also $\alpha : M \to \mathbb{R}_+ \setminus \{0\}$ be in $L^\infty(M) \cap L^1(M)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function with $f(0) = 0$ verifying (4.1.1) and $V : M \to \mathbb{R}$ be a potential verifying (V_1), (V_2). Assume that for some $a > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is non-increasing in $(0, a]$. Then, the following conditions are equivalent:*

(i) for each $b > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is not constant in $(0, b]$;

(ii) for each $r > 0$, there exists an open interval $I_r \subseteq (0, +\infty)$ such that for every $\lambda \in I_r$, problem $(\mathscr{P}_\lambda)$ has a nontrivial solution $u_\lambda \in H^1_g(M)$ satisfying

$$\int_M \left(|\nabla_g u_\lambda(x)|^2 + V(x)u_\lambda^2\right) dv_g < r.$$

We conclude the chapter with a corollary of the main result in the euclidean setting. We propose a more general set of assumption on V which implies both the compactness of the embedding of $H^1_V(\mathbb{R}^n)$ into and the discreteness of the spectrum of the Schrödinger operator, see Benci and Fortunato [8]. Namely, let $n \geq 3$, $\alpha : \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ be in $L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function with $f(0) = 0$ such that there exist two constants $k > 0$ and $q \in (1, 2^\star)$ such that

$$f(\xi) \leq k(1 + \xi^{q-1}) \text{ for all } \xi \geq 0.$$

Let also $V : \mathbb{R}^n \to \mathbb{R}$ be in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$, such that $\operatorname{essinf}_{\mathbb{R}^n} \mathrm{V} \equiv \mathrm{V}_0 > 0$ and

$$\int_{B(x)} \frac{1}{V(y)} dy \to 0 \qquad \text{as } |x| \to \infty,$$

where $B(x)$ denotes the unit ball in $\mathbb{R}^n$ centered at x. In particular, if V is a strictly positive ($\inf_{\mathbb{R}^n} V > 0$), continuous and coercive function, the above conditions hold true.

Corollary 4.1.1 (Faraci and Farkas [23]). *Assume that for some $a > 0$ the function $\xi \to \frac{F(\xi)}{\xi^2}$ is non-increasing in $(0, a]$. Then, the following conditions are equivalent:*

(i) for each $b > 0$, the function $\xi \to \frac{F(\xi)}{\xi^2}$ is not constant in $(0, b]$;

(ii) for each $r > 0$, there exists an open interval $I \subseteq (0, +\infty)$ such that for every $\lambda \in I$, problem

$$\begin{cases} -\Delta u + V(x)u = \lambda\alpha(x)f(u), & \text{in } \mathbb{R}^n \\ u \geq 0, & \text{in } \mathbb{R}^n \\ u \to 0, & \text{as } |x| \to \infty \end{cases}$$

has a nontrivial solution $u_\lambda \in H^1(\mathbb{R}^n)$ satisfying $\displaystyle\int_{\mathbb{R}^n} \left(|\nabla u_\lambda|^2 + V(x)u_\lambda^2\right) dx < r$.

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