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Intelligent Decision Models

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"To keep your balance, you must keep moving."

Albert Einstein

ÓBUDA UNIVERSITY

Abstract

Doctoral School of Applied Informatics and Applied Mathematics

Doctor of Philosophy

Intelligent Decision Models

by Orsolya CSISZÁR

This thesis is concerned with the development of intelligent decision models from a theoretical point of view. It covers two main topics: in chapters 2-3, two special types of aggregation functions are studied, while in chapters 4-6, the so-called general nilpotent operator system is introduced and examined.

Chapter 1 gives an introduction to the topic of aggregation functions and intelligent decision modeling. The basic preliminaries are also given here.

In Chapter 2, the properties of a new construction method of aggregation functions from two given ones, called threshold construction, are discussed. This class of non-symmetric functions provides a generalization of t-norms and t-conorms by partitioning the unit interval with respect to only one variable. In fuzzy modeling framework, the relationship of the input and the output can be modeled by splitting the input into fuzzy regions for which we can describe the output in different ways. In several applications, the roles of the inputs are not symmetric, which indicates the use of the examined construction. The results of this chapter can be found in Csiszár and Fodor, [19].

Chapter 3 presents new construction methods of uninorms with fixed values along the borders. Sufficient and necessary conditions are presented. The results of this chapter can be found in Csiszár and Fodor, [20].

In Chapter 4, the concept of a nilpotent connective system is introduced. It is shown that a consistent logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic, which means that nilpotent logical systems are wider than we have thought earlier. Using more than one generator functions, three naturally derived negations are examined. It is shown that the coincidence of the three negations leads back to a system which is isomorphic to Łukasiewicz-logic. Consistent nilpotent logical structures with three different negations are also provided. The results of this chapter can be found in Dombi and Csiszár, [27].

In Chapter 5, implication operators in bounded systems are deeply examined. Both R- and S-implications with respect to the three naturally derived negations of the bounded system are considered. It is shown that these implications never coincide in a bounded system, as the condition of coincidence is equivalent to the coincidence of the negations, which would lead to Łukasiewicz logic. The formulae and the basic properties of four different types of implications are given, two of which fulfill all the basic properties generally required for implications. The results of this chapter can be found in Dombi and Csiszár, [28].

Chapter 6 gives a detailed discussion of equivalence operators in bounded systems. Three different types of operators are studied. The paradox of the equivalence relation is solved by aggregating the implication-based equivalence and its dual operator. It is shown that the aggregated equivalence has preferable properties such as associativity and threshold transitivity. The results of this chapter can be found in Dombi and Csiszár, [29].

The final Chapter 7 summarizes the main results and suggests some future research directions that could provide the next steps along the path to a practical and widely applicable system.

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Contents

Declaration of Authorship **i**

Abstract **iii**

Acknowledgements **vi**

Contents **vii**

List of Figures **ix**

List of Tables **x**

Abbreviations **xi**

Symbols **xii**

1 Introduction **1**

1.1 Aggregation and decision 1

1.2 Basic preliminaries 3

1.2.1 Negations 3

1.2.2 Triangular norms and conorms 4

2 Threshold Construction of Aggregation Functions **8**

2.1 Motivation and scope 8

2.2 Properties 9

2.3 Symmetrization 10

2.4 Overview 14

3 Uninorms with Fixed Values along Their Borders **15**

3.1 Motivation and scope 15

3.2 Results 16

3.3 Overview 27

4 The General Nilpotent Operator System **29**

4.1 Motivation and scope 29

4.2 Characterization of strict negation operators 30

4.3 Nilpotent connective systems . . . 32
4.3.1 Structural properties of connective systems . . . 34
4.3.1.1 Classification property . . . 35
4.3.1.2 The De Morgan law . . . 36
4.3.2 Consistent connective systems . . . 41
4.3.2.1 Bounded systems . . . 43
4.4 Overview . . . 47

5 Implication Operators in Bounded Systems 49
5.1 Motivation and scope . . . 49
5.2 Preliminaries . . . 50
5.3 R-implications in bounded systems . . . 52
5.4 S-implications in bounded systems . . . 55
5.4.1 Properties of i_{S_n}, i_{S_d} and i_{S_c} . . . 56
5.4.2 S-implications and the ordering property . . . 57
5.5 Comparison of implications in bounded systems . . . 59
5.6 Min and Max operators in nilpotent connective systems . . . 60
5.7 Overview . . . 60

6 Equivalence Operators in Bounded Systems 64
6.1 Motivation and scope . . . 64
6.2 Preliminaries . . . 66
6.3 Equivalences in bounded systems . . . 67
6.3.1 Properties of $e_c(x, y)$ and $e_d(x, y)$ . . . 68
6.4 Dual equivalences . . . 71
6.4.1 Properties of $\bar{e}_d$ and $\bar{e}_c$ . . . 72
6.5 Arithmetic mean operators in bounded systems . . . 74
6.6 Aggregated equivalences . . . 75
6.6.1 Properties of the aggregated equivalence operator . . . 77
6.7 Applications . . . 81
6.8 Overview . . . 81

7 Main Results and Further Work 85

A Tables 88

Bibliography 92

List of Figures

1.1 Continuous negations with different ν_* values 4

2.1 Special types of $A_{<T,a,S>}(x,y)$ 9
2.2 Special types of $B_{<T,a,S>}(x,y)$ 11
2.3 Special types of $C_{<T,a,S>}(x,y)$ and $D_{<T,a,S>}(x,y)$ 12

3.1 U_1 and U_2 19
3.2 U_3 and U_4 22
3.3 U_5 22
3.4 U_5' and U_5'' 24
3.5 U_6 25
3.6 U_6' and U_6'' 26

4.1 $n_d(x) < n(x) < n_c(x)$ for $\alpha = 2$ 44
4.2 The relationship between ν, ν_c and ν_d in consistent rational systems . . . 46
4.3 Conjunction $c[x,y]$ and disjunction $d[x,y]$ 48

5.1 i_c and i_d implications for rational generators 61
5.3 S_c-implications for rational generators 62
5.2 S_n-implications for rational generators 62

6.1 $e_c(x,y)$ and $e_d(x,y)$ for rational generators 68
6.2 $\bar{e}_c(x,y)$ and $\bar{e}_d$ with rational generators 72
6.3 The domain of aggregated equivalences 77
6.4 Aggregated equivalences with rational generators with $\nu = 0.3$ 77
6.5 Pointwise equivalence of fuzzy numbers with rational generators ($\nu_c = \nu_d = 0.3$) 82
6.6 Pointwise dual equivalence of fuzzy numbers with rational generators ($\nu_c = \nu_d = 0.3$) 83
6.7 Pointwise aggregated equivalence of triangular fuzzy numbers with rational generators ($\nu = 0.6$) 84

List of Tables

A.1 Power functions as normalized generators 88
A.2 Power functions as normalized generators – logical connectives 88
A.3 Exponential functions as normalized generators 89
A.4 Rational functions as normalized generators 89
A.5 Rational functions as normalized generators – 3 negations 89
A.6 Mixed types of normalized generator functions 90
A.7 Properties of implications in bounded systems 90
A.8 The main properties of equivalence operators 91

Abbreviations

BC	**B**oundary **C**ondition
LC	**L**aw of **C**ontraposition
DF	**D**ominance of **F**alsity
DT	**D**ominance of **T**ruth
EP	**E**xchange **P**rinciple
FA	**F**irst Place **A**ntitonicity
IP	**I**dentity **P**rinciple
LN	**L**eft **N**eutrality Property
OP	**O**rdering **P**roperty
SI	**S**econd Place **I**sotonicity
SN	**S**trong **N**egation Property

Symbols

$c(x,y)$	conjunction
$d(x,y)$	disjunction
e	neutral element
$e(x,y)$	equivalence
$e_c(x,y)$	equivalence given by 6.7
$e_d(x,y)$	equivalence given by 6.7
$\bar{e}_c(x,y)$	dual equivalence given by 6.15
$\bar{e}_d(x,y)$	dual equivalence given by 6.15
$e_c^*(x,y)$	aggregated equivalence given by 6.23
$e_d^*(x,y)$	aggregated equivalence given by 6.23
$f_c(x)$	normalized additive generator function of a conjunction c
$f_d(x)$	normalized additive generator function of a disjunction d
$f_n(x)$	generator function of a negation n
$i(x,y)$	implication
$i_c(x,y)$	implication given by 5.5
$i_d(x,y)$	implication given by 5.5
$i_R(x,y)$	residual implication
$i_S(x,y)$	S-implication
$m_c^\alpha(x,y)$	weighted arithmetic mean operator given by 6.21
$m_c^\alpha(x,y)$	weighted arithmetic mean operator given by 6.21
$n(x)$	negation
$n_c(x)$	negation generated by f_c
$n_d(x)$	negation generated by f_d
S	t-conorm
S_D	drastic sum

S_L	Łukasiewicz t-conorm
S_M	maximum
S_P	probabilistic sum
$s(x)$	additive generator function of a t-conorm
T	t-norm
T_D	drastic product
T_L	Łukasiewicz t-norm
T_M	minimum
T_P	product
$t(x)$	additive generator function of a t-norm
U	uninorm
U_{min}	uninorm given by 3.1
U_{max}	uninorm given by 3.2
ν	neutral value of a negation
ν_c	neutral value of the negation generated by f_c
ν_d	neutral value of the negation generated by f_d
$[\ ]$	cutting function given by 1.9

Dedicated to ϱ

Chapter 1

Introduction

1.1 Aggregation and decision

Aggregation is the process of combining several numerical values into a single representative one. The function, which performs this process is called an aggregation function. Despite the simplicity of this definition, the size of the field of its applications is incredibly huge: applied mathematics (e.g. probability theory, statistics, decision theory), computer sciences (e.g. artificial intelligence, operation research, pattern recognition and image processing), economics and finance, multicriteria decision aid, etc (see e.g. [10], [43]).

If we think of the arithmetic mean, we can see that the history of aggregation is as old as mathematics itself. However, it was only in the last decades, when the rapid development of the above mentioned fields (mainly due to the arrival of computers) made it necessary to establish a sound theoretic basis for aggregation. The problem of data fusion, synthesis of information or aggregating criteria to form overall decision is of considerable importance in many fields of human knowledge. Due to the fact that data is obtained in an easier way, this field is of increasing interest.

One of the most prominent group of applications of aggregation functions comes from decision theory. Making decisions often leads to aggregating preferences or scores on a given set of alternatives, the preferences being obtained from several decision makers, experts, voters or representing different points of view, criteria, objectives. This concerns decision under multiple criteria or multiple attributes, multiperson decision making and multiobjective optimization [38].

The main factor in determining the structure of the needed aggregation function is the relationship between the criteria. At one extreme there is the case in which we desire all the criteria to be satisfied. At the other extreme is the situation in which we want the satisfaction of any of the criteria. These two extreme cases lead to the use of "and" and "or" operators to combine the criteria functions. A decision can be interpreted as the intersection of fuzzy sets, usually computed by applying a t-norm based operator, when there is no compensation between low and high degrees of membership. If it is interpreted as the union of fuzzy sets, represented by a t-conorm based operator, full compensation is assumed. However, it is obvious that no managerial decision represents any of these extreme situations.

As it is well-known, uninorms generalize both t-norms and t-conorms as they allow a neutral element anywhere in the unit interval. In the first part of the thesis (sections 2 and 3), results on further constructions of continuous aggregation functions are presented.

Another outstanding application of aggregation functions comes from artificial intelligence, fuzzy logic [34]. Pattern recognition and classification, as well as image analysis are typical examples. According to Aristotle, in mathematics it was originally assumed that "the same thing cannot at the same time both belong and not belong to the same object and in the same respect. [...] Of any object, one thing must be either asserted or denied." The idea of many-valued logic was initiated by Jan Łukasiewicz around 1920. "Logic changes from its very foundations if we assume that in addition to truth and falsehood there is also some third logical value or several such values" [50]. Many-valued logic was for several decades considered as a purely theoretical topic. It was the introduction of fuzzy sets by Zadeh in 1965 [82], which opened the way to fuzzy logics.

Aggregation functions are inevitably used in fuzzy logic, as a generalization of logical connectives. In artificial intelligence, these techniques are mainly used when a system has to make a decision. It is possible that the system has not only a single criteria for each alternative, but several ones. This case corresponds to a multicriteria decision-making problem. Furthermore, if a system needs a good representation of an environment, it needs the knowledge supplied by information sources in order to be reliable. However, the information supplied by a single information source (by a single expert or sensor) is often not reliable enough. That is why the information provided from several sensors (or experts) should be combined to improve data reliability and accuracy and also to include some features that are impossible to perceive with individual sensors.

One of the most significant problems of fuzzy set theory is the proper choice of set-theoretic operations [68, 77]. The class of nilpotent t-norms has preferable properties which make them more usable in building up logical structures. Among these properties are the fulfillment of the law of contradiction and the excluded middle, or the coincidence of the residual and the S-implication [33, 74]. Due to the fact that all continuous

Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [43] [50], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic.

In the second part of the thesis (sections 4, 5 and 6), logical systems, more specifically, nilpotent logical systems are in consideration. It is shown that a consistent logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic. This new type of nilpotent logical systems is called a bounded system, which has the advantage of three naturally derived negations. Implication and equivalence operators in bounded systems are deeply examined and a wide range of examples is also presented.

1.2 Basic preliminaries

First, I recall some basic notations and results regarding negation operators, t-norms and t-conorms that will be useful in the sequel.

1.2.1 Negations

Definition 1.1. A unary operation $n : [0,1] \to [0,1]$ is called a negation if it is non-increasing and compatible with classical logic, i.e. $n(0) = 1$ and $n(1) = 0$.
A negation is strict if it is also strictly decreasing and continuous.
A negation is strong if it is also involutive, i.e. $n(n(x)) = x$.

Due to the continuity and strict monotonicity of n, for continuous negations there always exists some ν_*, for which $n(\nu_*) = \nu_*$ holds. ν_* is called the *neutral value* of the negation and the notation n_{ν_*} stands for a negation operator with neutral value ν_*. In the literature ν_* is often denoted by e. In Figure 1.1 we can see some negations with different ν_* values.

Drastic negations [78] are the so-called intuitionistic and dual intuitionistic negations (denoted by n_0 and n_1 respectively):

$$n_0(x) = \begin{cases} 1 & if \quad x = 0 \\ 0 & if \quad x > 0 \end{cases} \quad \text{and} \quad n_1(x) = \begin{cases} 1 & if \quad x < 1 \\ 0 & if \quad x = 1 \end{cases}$$

These drastic negations are neither continuous nor strictly decreasing, therefore they are not strict negations, but we can get them as limits of strict negations.

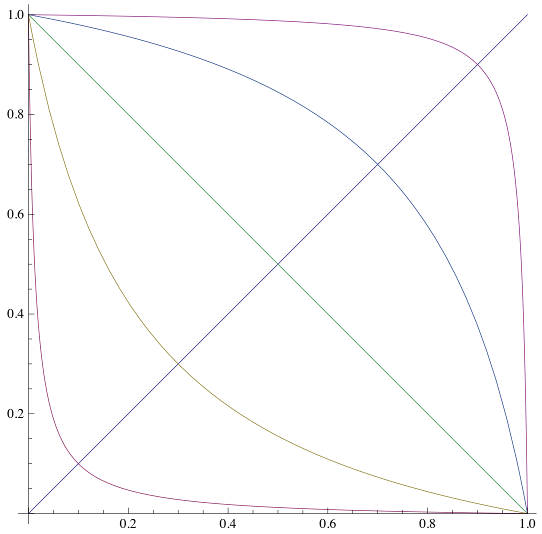


FIGURE 1.1: Continuous negations with different ν_* values

Definition 1.2. A continuous, strictly increasing function $\varphi : [a, b] \to [a, b]$ with boundary conditions $\varphi(a) = a, \varphi(b) = b$ is called an automorphism of $[a, b]$.

The well-known representation theorem was obtained by Trillas.

Proposition 1.3. *(Trillas, [73]) n is a strong negation if and only if*

$$n(x) = f_n^{-1}(1 - f_n(x)),$$

where $f_n : [0, 1] \to [0, 1]$ is an automorphism of $[0, 1]$.

Remark 1.4. This result also means that n is a strong negation iff

$$n(x) = f_n^{-1}\left(n'\left(f_n(x)\right)\right) \tag{1.1}$$

where f_n, called the generator function of n, $f_n : [0; 1] \to [0; 1]$ is a strictly monotone, continuous function with $f_n(0) = 0$ and $f_n(1) = 1$ and n' is a strong negation.

Example 1.1. *For $f_n(x) = x^2$ and $n'(x) = \frac{1-x}{1+x}$ we get $n(x) = \sqrt{\frac{1-x^2}{1+x^2}}$.*

1.2.2 Triangular norms and conorms

A triangular norm (*t-norm* for short) T is a binary operation on the closed unit interval $[0, 1]$ such that $([0, 1], T)$ is an abelian semigroup with neutral element 1 which is totally ordered, i.e., for all x_1, x_2, y_1, $y_2 \in [0, 1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ we have $T(x_1, y_1) \leq T(x_2, y_2)$, where $\leq$ is the natural order on $[0, 1]$.

Standard examples of t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$, and the drastic product $T_{\mathbf{D}}$ with $T_{\mathbf{D}}(1,x) = T_{\mathbf{D}}(x,1) = x$, and $T_{\mathbf{D}}(x,y) = 0$ otherwise. Clearly, $T_{\mathbf{M}}$ and $T_{\mathbf{D}}$ are the greatest and smallest t-norm, respectively, i.e., for each t-norm T we have $T_{\mathbf{D}} \leq T \leq T_{\mathbf{M}}$.

A triangular conorm (*t-conorm* for short) S is a binary operation on the closed unit interval $[0,1]$ such that $([0,1],S)$ is an abelian semigroup with neutral element 0 which is totally ordered. Standard examples of t-conorms are the maximum $S_{\mathbf{M}}$, the probabilistic sum $S_{\mathbf{P}}$, the Łukasiewicz t-conorm $S_{\mathbf{L}}$ given by $S_{\mathbf{L}}(x,y) = \min(x+y,1)$, and the drastic sum $S_{\mathbf{D}}$ with $S_{\mathbf{D}}(0,x) = S_{\mathbf{D}}(x,0) = x$, and $S_{\mathbf{D}}(x,y) = 1$ otherwise. Clearly, $S_{\mathbf{M}}$ and $S_{\mathbf{D}}$ are the smallest and greatest t-conorm, respectively, i.e., for each t-cnorm S we have $S_{\mathbf{M}} \leq S \leq S_{\mathbf{D}}$.

A continuous t-norm T is said to be *Archimedean* if $T(x,x) < x$ holds for all $x \in (0,1)$. A continuous Archimedean T is called *strict* if T is strictly monotone; i.e. $T(x,y) < T(x,z)$ whenever $x \in (0,1]$ and $y < z$, and *nilpotent* if there exist $x,y \in (0,1)$ such that $T(x,y) = 0$.

From the duality between t-norms and t-conorms, we can easily derive the following properties. A continuous t-conorm S is said to be *Archimedean* if $S(x,x) > x$ holds for every $x,y \in (0,1)$. A continuous Archimedean S is called *strict* if S is strictly monotone; i.e. $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$, and *nilpotent* if there exist $x,y \in (0,1)$ such that $S(x,y) = 1$. A t-norm is said to be *positive* if $x,y > 0$ implies $T(x,y) > 0$.

From the duality between t-norms and t-conorms we can easily get the following properties as well. A continuous t-conorm S is said to be *Archimedean* if $S(x,x) > x$ holds for every $x,y \in (0,1)$, *strict* if S is strictly monotone i.e. $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$, and *nilpotent* if there exist $x,y \in (0,1)$ such that $S(x,y) = 1$.

Proposition 1.5. *(Baczyński, [7], Ling, [53]) A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm iff it has a continuous additive generator, i.e. there exists a continuous strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(min(t(x)+t(y),t(0)), \quad x,y \in [0,1]. \tag{1.2}$$

Proposition 1.6. *(Baczyński, [7], Ling, [53]) A function $S : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-conorm iff it has a continuous additive generator, i.e. there exists a continuous strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely* *determined up to a positive multiplicative constant, such that*

$$S(x,y) = s^{-1}(min(s(x)+s(y),s(1)), \quad x,y \in [0,1]. \tag{1.3}$$

Proposition 1.7. *[43]*

A t-norm T is strict if and only if $t(0) = \infty$ holds for each continuous additive generator t of T.

A t-norm T is nilpotent if and only if $t(0) < \infty$ holds for each continuous additive generator t of T.

A t-conorm S is strict if and only if $s(1) = \infty$ holds for each continuous additive generator s of S.

A t-conorm S is nilpotent if and only if $s(1) < \infty$ holds for each continuous additive generator s of S.

In both of the above mentioned Propositions 1.5 and 1.6 we can allow the generator functions to be strictly increasing *or* strictly decreasing, which will result in the fact that they will be determined up to a (not necessarily positive) multiplicative constant. For an increasing generator function t of a t-conorm and similarly for a decreasing generator function s of a t-conorm, *min* in (1.2) and (1.3) has to be replaced by *max*. In this case we will have $t(0) = \pm\infty$ and $s(1) = \pm\infty$ for strict norms and similarly, $t(0) < \infty$ *or* $t(0) > -\infty$ and $s(1) < \infty$ *or* $s(1) > -\infty$ for the nilpotent ones.

Proposition 1.8. *[43] Let T be a continuous Archimedean t-norm.*

If T is strict, then it is isomorphic to the product t-norm $T_{\mathbf{P}}$, i.e., there exists an automorphism of the unit interval ϕ such that $T_\phi = \phi^{-1}\left(T(\phi(x),\phi(y))\right) = T_{\mathbf{P}}$.

If T is nilpotent, then it is isomorphic to the Łukasiewicz t-norm $T_{\mathbf{L}}$, i.e., there exists an automorphism of the unit interval ϕ such that $T_\phi = \phi^{-1}\left(T(\phi(x),\phi(y))\right) = T_{\mathbf{L}}$.

From the definitions of t-norms and t-conorms it follows immediately that t-norms are conjunctive, while t-conorms are disjunctive aggregation functions. Therefore, they are widely used as conjunctions and disjunctions in multivalued logical structures.

The logical system based on the nilpotent Łukasiewicz t-norm as conjunction is called *Lukasiewicz-logic* [45, 54, 63].

The use of the so-called cutting function makes the formulae simpler.

Definition 1.9. (Sabo et al., [66], Dombi and Csiszár, [27]) Let us define the cutting operation [ ] by

$$[x] = \begin{cases} 0 & if \quad x < 0 \\ x & if \quad 0 \leq x \leq 1 \\ 1 & if \quad 1 < x \end{cases}$$

and let the notation [ ] also act as 'brackets' when writing the argument of an operator, so that we can write $f[x]$ instead of $f([x])$.

Chapter 2

Threshold Construction of Aggregation Functions

2.1 Motivation and scope

In 2012 Massanet and Torrens [58] introduced a new construction method of a fuzzy implication from two given ones. Now this idea is appropriately extended to constructions of continuous aggregation functions from a t-norm and a t-conorm, based on an adequate scaling on the second variable of the initial operators [19]. This construction can be usuful in fuzzy applications where the inputs have different semantic contents.

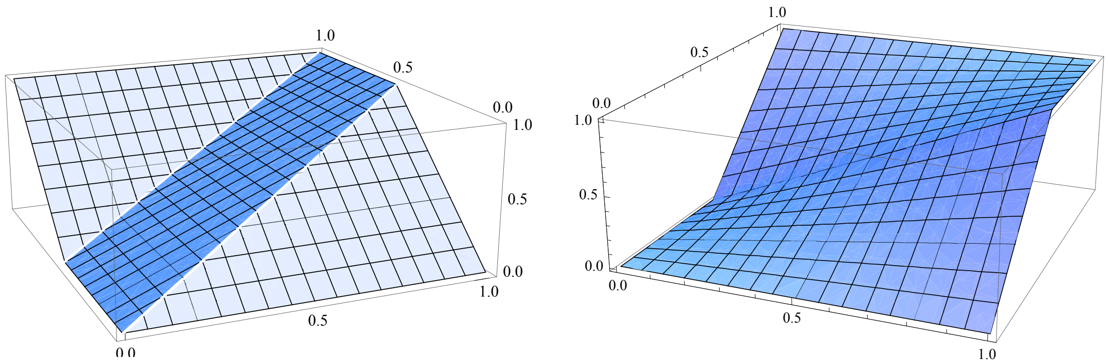
Let T be a t-norm, S be a t-conorm, and $a \in (0,1)$. Let us define two binary operators as follows. Let

$$A_{<T,a,S>}(x,y) = \begin{cases} T(x, \frac{y}{a}), & \text{if } \ x \in [0,1], y \in [0,a] \\ S\left(x, \frac{y-a}{1-a}\right), & \text{if } \ x \in [0,1], y \in (a,1] \end{cases},$$

and

$$A^{<T,a,S>}(x,y) = A_{<T,a,S>}(y,x).$$

See Fig. 2.1.



(a) $A_{\langle T_{\mathbf{M}}, \frac{1}{2}, S_{\mathbf{M}} \rangle}$ (b) $A_{\langle T_{\mathbf{P}}, \frac{1}{2}, S_{\mathbf{P}} \rangle}$

FIGURE 2.1: Special types of $A_{<T,a,S>}(x,y)$

2.2 Properties

In this section we examine the main properties, such as neutral or idempotent elements, associativity, of $A_{<T,a,S>}(x,y)$ and $A^{<T,a,S>}(x,y)$.

As it is easy to see, a is the *right neutral element* of $A_{<T,a,S>}(x,y)$, i.e., $A_{<T,a,S>}(x,a) = T(x,1) = x$, for all $x \in [0,1]$. Similarly, $A^{<T,a,S>}(x,1) = 1$ and $A^{<T,a,S>}(x,0) = 0$.

We can also see that a is an *idempotent element* of $A_{<T,a,S>}$ and $A^{<T,a,S>}$.

It can be shown that $A_{<T,a,S>}(x,y)$ and $A^{<T,a,S>}(x,y)$ are always *monotone non-decreasing.*

Due to the construction, both functions $A_{<T,a,S>}(x,y)$ and $A^{<T,a,S>}(x,y)$ are continuous when T and S are continuous.

Since associativity is a key property of t-norms and t-conorms, it would be good to hand down some version of associativity to the constructions. A simple counterexample reveals that neither $A_{<T,a,S>}(x,y)$ nor $A^{<T,a,S>}(x,y)$ is associative for arbitrary T and S. Indeed, let $T = T_{\mathbf{M}}$, $S = S_{\mathbf{M}}$, and $a = \frac{1}{2}$. Then we have

$$A_{<T,a,S>}\left(A_{<T,a,S>}\left(\frac{2}{3},\frac{2}{3}\right),\frac{1}{4}\right) = \frac{1}{2} \neq A_{<T,a,S>}\left(\frac{2}{3},A_{<T,a,S>}\left(\frac{2}{3},\frac{1}{4}\right)\right) = \frac{2}{3}.$$

Thus, we study the following problem. Let T be a given t-norm. Does there exist a t-conorm S such that $A^{<T,a,S>}$ is associative?

To solve the functional equation of associativity, the following cases are to be examined:

1. $y \leq a, z \leq a$

2. $y \leq a, z > a$

3. $y > a, z \leq a$

4. $y > a, z > a$.

It can be shown that if A is associative, then T is necessarily a-migrative, i.e., $T(a, v) = av \ \forall v \in [0, 1]$, see the definition and fundamental results about migrativity in [39].

In a similar way, we can conclude that $S(a, v) = a + (1 - a)v \ \forall v \in [0, 1]$.

This means, that the a-migrativity is necessary but not sufficient condition of associativity of A. Indeed, for example when $T = T_{\mathbf{P}}$, the constructed A will not be associative. Moreover, if we examine cases *iii.* and *iv.*, we can see that A can never be associative on the whole unit square $[0, 1]$.

Studying the particular case when $T = T_{\mathbf{P}}$, $SS_{\mathbf{P}}$, and $a = \frac{1}{2}$, we obtain

$$A_{<T_{\mathbf{P}}, \frac{1}{2}, S_{\mathbf{P}}>}(x, y) \quad = \quad \begin{cases} 2xy, & \text{if } \ x \in [0,1], y \in \left[0, \frac{1}{2}\right] \\ 2x + 2y - 2xy - 1, & \text{if } \ x \in [0,1], \ y \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

It can easily be seen that $A_{<T_{\mathbf{P}}, \frac{1}{2}, S_{\mathbf{P}}>}(x, y)$ is not associative. Moreover, $A_{<T_{\mathbf{P}}, \frac{1}{2}, S_{\mathbf{P}}>}(x, y)$ does not fulfill any of the associativity-like equations (see e.g. Grassmann, cyclic, Hosszú in [56]), and it is not bisymmetric either.

2.3 Symmetrization

Since $A_{<T,a,S>}(x, y)$ and $A^{<T,a,S>}(x, y)$ are obviously not symmetric, it is natural to study symmetrized versions of both functions. First we consider the following way of symmetrization of $A_{<T,a,S>}$, denoted by $B_{<T,a,S>}$:

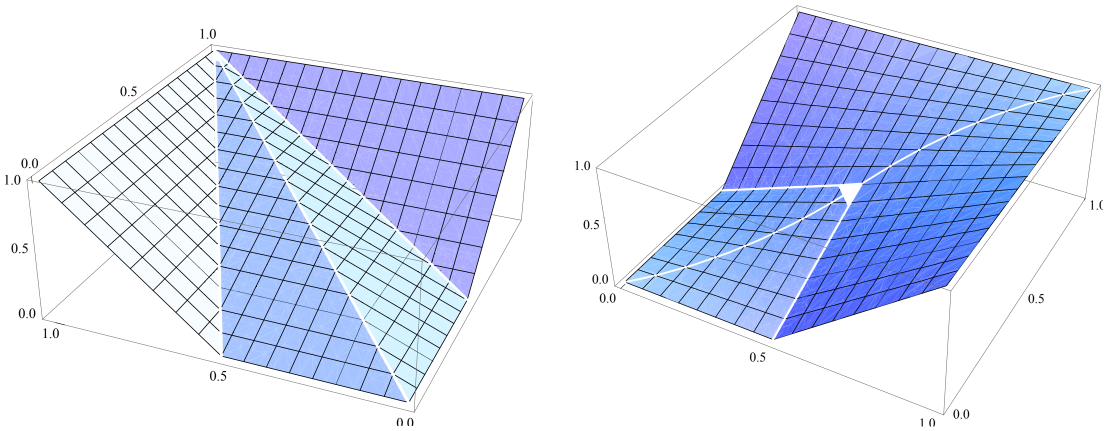
$$B_{<T,a,S>}(x, y) = A_{<T,a,S>}(\min(x, y), \max(x, y)).$$

See Fig. 2.2.

The monotonicity of $B_{<T,a,S>}(x, y)$ follows from the monotonicity of $A_{<T,a,S>}(x, y)$.

The continuity of $B_{<T,a,S>}$ follows from the continuity of $A_{<T,a,S>}$ and $A^{<T,a,S>}$.

We can see that $B_{<T,a,S>}(x, y)$ is not associative for arbitrary T and S either.



(a) $B_{\langle T_{\mathbf{M}},\frac{1}{2},S_{\mathbf{M}}\rangle}$ (b) $B_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y)$

FIGURE 2.2: Special types of $B_{<T,a,S>}(x,y)$

Moreover, $B_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y)$ does not fulfil any of the associativity-like equations and it is not bisymmetric.

Let us study the particular case when $T = T_{\mathbf{P}}$, $S = S_{\mathbf{P}}$, and $a = \frac{1}{2}$. In this case we obtain

$$B_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y) = \begin{cases} 2xy, & \text{if } \max(x,y) \in \left[0,\frac{1}{2}\right] \\ 2x+2y-2xy-1, & \text{if } \max(x,y) \in \left(\frac{1}{2},1\right]. \end{cases}$$

Now we define two further symmetrization of $A_{<T,a,S>}$, denoted by $C_{<T,a,S>}$ and $D_{<T,a,S>}$, as follows:

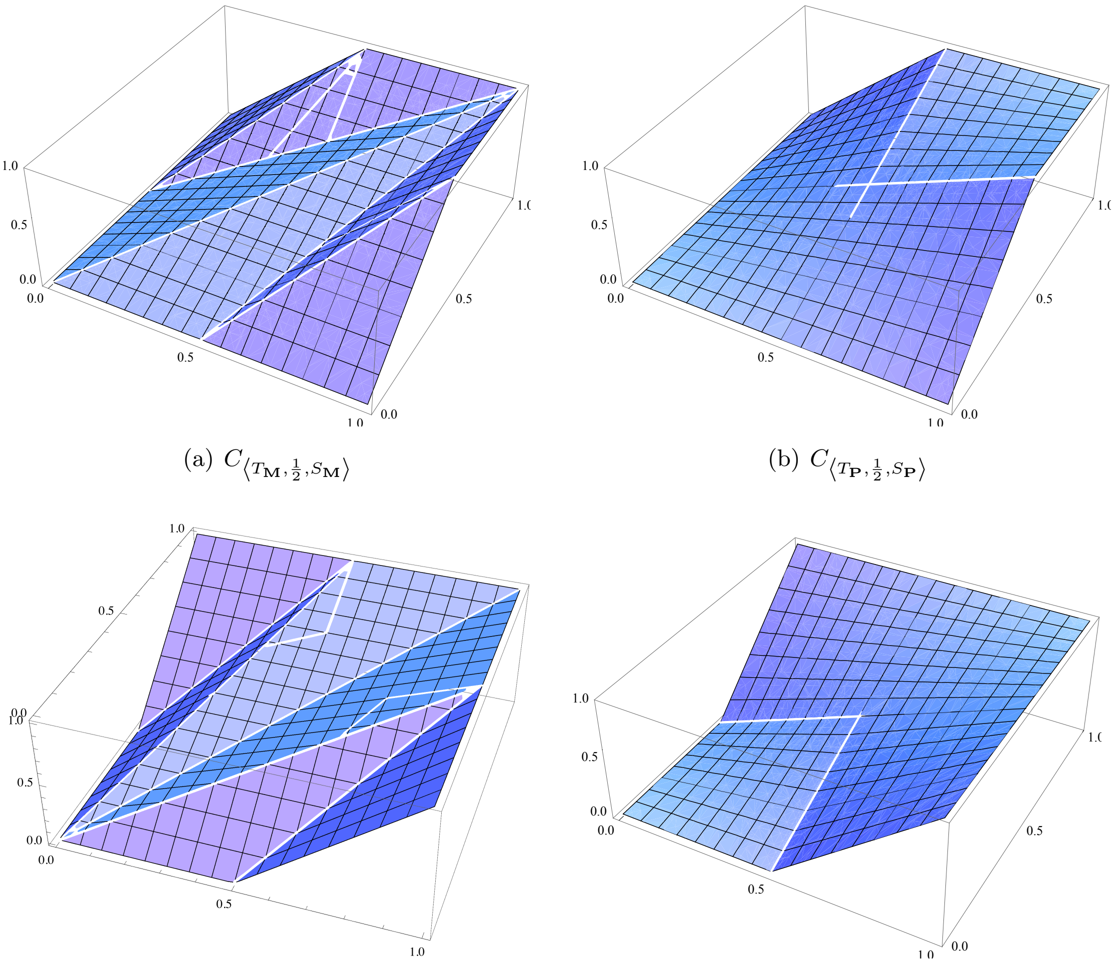
$$C_{<T,a,S>}(x,y) = \min\left(A_{<T,a,S>}(x,y), A^{<T,a,S>}(x,y)\right)$$

and

$$D_{<T,a,S>}(x,y) = \max\left(A_{<T,a,S>}(x,y), A^{<T,a,S>}(x,y)\right).$$

See Fig. 2.3. Both functions can be written in explicit forms, by applying the definitions of $A_{<T,a,S>}$:

$$C_{<T,a,S>}(x,y) = \begin{cases} S\left(x, \frac{y-a}{1-a}\right), & \text{if } x \in (a,1], y \in (a,1] \\ T\left(x, \frac{y}{a}\right) & \text{otherwise,} \end{cases}$$



(a) $C_{\langle T_{\mathbf{M}}, \frac{1}{2}, S_{\mathbf{M}} \rangle}$

(b) $C_{\langle T_{\mathbf{P}}, \frac{1}{2}, S_{\mathbf{P}} \rangle}$

(c) $D_{\langle T_{\mathbf{M}}, \frac{1}{2}, S_{\mathbf{M}} \rangle}$

(d) $D_{\langle T_{\mathbf{P}}, \frac{1}{2}, S_{\mathbf{P}} \rangle}$

FIGURE 2.3: Special types of $C_{<T,a,S>}(x,y)$ and $D_{<T,a,S>}(x,y)$

and

$$D_{<T,a,S>}(x,y) = \begin{cases} T\left(x, \frac{y}{a}\right), & \text{if } \ x \in [0,a], y \in [0,a] \\ S\left(x, \frac{y-a}{1-a}\right) & \text{otherwise.} \end{cases}$$

We can see that $C_{<T,a,S>}(1,x) = 1$ if $x \in (a,1]$, and similarly, $C_{<T,a,S>}(0,x) = 0$.

For D we obtain $D_{<T,a,S>}(0,x) = 0$ if $x \leq a$, and in this case $D_{<T,a,S>}(0,x) = 0$ also holds.

Clearly, $C_{<T,a,S>}(x,y)$ and $D_{<T,a,S>}(x,y)$ are monotone non-decreasing.

The continuity of $C_{<T,a,S>}$ and $D_{<T,a,S>}$ follows from the continuity of $A_{<T,a,S>}$ and $A^{<T,a,S>}$.

However, it can be shown that $C_{<T,a,S>}(x,y)$ is not associative.

Moreover, $C_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y)$ does not fulfil any of the associativity-like equations and it is not bisymmetric.

Studying the particular case when $T = T_{\mathbf{P}}$, $S = S_{\mathbf{P}}$, and $a = \frac{1}{2}$, we obtain

$$C_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y) = \begin{cases} 2xy, & \text{if } x \in \left[0,\frac{1}{2}\right], y \in \left[0,\frac{1}{2}\right] \\ 2x+2y-2xy-1, & \text{if } x \in \left(\frac{1}{2},1\right], y \in \left(\frac{1}{2},1\right] \\ \min\left(2xy, 2x+2y-2xy-1\right), & \text{if } x \in \left[0,\frac{1}{2}\right], y \in \left(\frac{1}{2},1\right] \\ \min\left(2xy, 2x+2y-2xy-1\right), & \text{if } x \in \left(\frac{1}{2},1\right], y \in \left[0,\frac{1}{2}\right], \end{cases}$$

that is,

$$C_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y) = \begin{cases} 2x+2y-2xy-1, & \text{if } x \in \left(\frac{1}{2},1\right], y \in \left(\frac{1}{2},1\right] \\ 2xy, & \text{otherwise.} \end{cases}$$

Similarly, we have

$$D_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y) = \begin{cases} 2xy, & \text{if } x \in \left[0,\frac{1}{2}\right], y \in \left[0,\frac{1}{2}\right] \\ 2x+2y-2xy-1, & \text{if } x \in \left(\frac{1}{2},1\right], y \in \left(\frac{1}{2},1\right] \\ \max\left(2xy, 2x+2y-2xy-1\right), & \text{if } x \in \left[0,\frac{1}{2}\right], y \in \left(\frac{1}{2},1\right] \\ \max\left(2xy, 2x+2y-2xy-1\right), & \text{if } x \in \left(\frac{1}{2},1\right], y \in \left[0,\frac{1}{2}\right], \end{cases}$$

i.e.

$$D_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y) = \begin{cases} 2xy, & \text{if } x \in \left[0,\frac{1}{2}\right], y \in \left[0,\frac{1}{2}\right] \\ 2x+2y-2xy-1, & \text{otherwise.} \end{cases}$$

Finally, using a representable uninorm U and its dual U^d instead of min and max in the process of symmetrization we can define another functions $E_{<T,a,S>}(x,y)$ and $F_{<T,a,S>}(x,y)$ as follows:

$$E_{<T,a,S>}(x,y) = U\left(A_{<T,a,S>}(x,y), A^{<T,a,S>}(x,y)\right)$$

and

$$F_{<T,a,S>}(x,y) = A_{<T,a,S>}\left(U(x,y), U^d(x,y)\right).$$

In the particular case when U is the so-called 3Π operator, ie., when

$$U(x,y) = \begin{cases} 0 & \text{, if } (x,y) \in \{(1,0),(0,1)\} \\ \dfrac{xy}{(1-x)(1-y)+xy} & \text{, otherwise,} \end{cases}$$

the symmetrized function $F_{<T_{\mathbf{P}},\frac{1}{2},S_{\mathbf{P}}>}(x,y)$, does not fulfil any of the associativity-like equations and it is not bisymmetric either.

2.4 Overview

In this section, a new generation method of aggregation functions from two given ones, called threshold construction method, was examined. The most usual properties were investigated and the necessary conditions to ensure them were studied.

Thesis 1.1. The new type of aggregation function turned out to be monotonic and continuous, having a right-neutral and idempotent element. Three possible ways of symmetrizations are studied, two of them using min-max operators and the third using uninorms. After proving the lack of associativity in all cases, the bisymmetry and all the other associativity-like equations known from the literature are studied.

Chapter 3

Uninorms with Fixed Values along Their Borders

3.1 Motivation and scope

The concept of uninorms was introduced in 1996 by Yager and Rybalov [79], as a generalization of both t-norms and t-conorms (see also Dombi, [31]). Since their introduction, uninorms have been studied deeply by numerous authors from theoretical and also from application points of view. They turned out to be useful in many fields like expert systems [22], aggregation [80] and fuzzy integral [11, 49]. Idempotent uninorms were characterized in [21]. Recently, a characterization of the class of uninorms with strict underlying t-norm and t-conorm was presented in [41]. In [52] the authors showed that uninorms with nilpotent underlying t-norm and t-conorm belong to U_{min} or U_{max}. In this section some further construction methods of uninorms from given t-norms and t-conorms are discussed and sufficient and necessary conditions are presented.

The results of this section can be found in Csiszár and Fodor, [20].

Definition 3.1. A mapping $U : [0,1] \times [0,1] \to [0,1]$ is a uninorm if it is commutative, associative, nondecreasing and there exists $e \in [0,1]$ such that $U(e,x) = x$ for all $x \in [0,1]$.

The structure of uninorms was first examined by Fodor, Yager and Rybalov in [40]. First let us recall two classes of uninorms from [40] that play a key role in this section.

Proposition 3.2. *Suppose that U is a uninorm with neutral element $e \in]0,1[$ and both functions $x \to U(x,1)$ and $x \to U(x,0)$ $(x \in [0,1])$ are continuous except perhaps at the point $x = e$. Then U is given by one of the following forms.*

1. *If* $U(0,1) = 0$ *then*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ min(x,y), & otherwise. \end{cases} \tag{3.1}$$

2. *If* $U(0,1) = 1$ *then*

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ max(x,y), & otherwise. \end{cases} \tag{3.2}$$

The class of uninorms having form 3.1 is denoted by U_{min}*, while the class with form 3.2 is denoted by* U_{max}*.*

3.2 Results

Proposition 3.3. *(See also Li et al., [52].) Let T be a strict t-norm, S be a strict t-conorm and $e \in]0,1[$. The function*

$$U_1(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1 \text{ or } y = 1; \\ min(x,y), & otherwise. \end{cases} \tag{3.3}$$

is a uninorm with neutral element e (see Figure 3.1).

Proof. To prove that U_1 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0,1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0,1)$ is a neutral element.

Note that U_1 differs from U_{min} only at points where either $x = 1$ or $y = 1$. Since the associativity of U_{min} is already known from [40], we only need to concentrate on the border lines where at least one of the variables of U_1 is 1.

To examine the associative equation

$$U_1(x, U_1(y,z)) = U_1(U_1(x,y), z), \tag{3.4}$$

we need to take the following possibilities into consideration:

1. If $x = 1$ or $z = 1$, then $U_1(x, U_1(y, z)) = 1 = U_1(U_1(x, y), z)$.
2. If $U_1(y, z) = 1$ or $U_1(x, y) = 1$, then by using the strict monotonicity of $S(x, y)$, we get $x = 1$ or $y = 1$. Thus $U_1(x, U_1(y, z)) = 1 = U_1(U_1(x, y), z)$.

$\square$

Remark 3.4. Note that the strict property of S cannot be omitted in Proposition 3.3 (i.e. the statement does not hold for arbitrary t-conorms). For a counterexample let us choose $T_{\mathbf{P}}$, $S_{\mathbf{L}}$, $e = 0.3$, $x = 0.7$, $y = 0.8$, and $z = 0$. In this case $U_1(0.7, U_1(0.8, 0)) = 0$, while $U_1(U_1(0.7, 0.8), 0) = 1$.

Proposition 3.5. *(See also Theorem 4 in Li et al., [52].) U_1 in (3.3) is a uninorm if and only if S is dual to a positive t-norm.*

Proof. The condition is sufficient, since in this case the proof is similar to that of Proposition 3.3.

Now I show that it is also necessary.

Let us assume indirectly that there exist $x_0, y_0 \neq 1$, for which $U_1(x_0, y_0) = 1$. Obviously, $x_0, y_0 > e$. Let $z_0 < e$, $z_0 \neq 1$ so that $U(y_0, z_0) \neq 1$. In this case the right hand side of the associativity equation in (3.4) is trivially 1, while the left hand side is z_0, which is a contradiction. $\square$

Proposition 3.6. *Let T be a strict t-norm, S be a strict t-conorm and $e \in]0, 1[$. The function*

$$U_2(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1, y \neq 0 \text{ or } x \neq 0, y = 1; \\ min(x,y), & otherwise. \end{cases} \tag{3.5}$$

is a uninorm with neutral element e (see Figure 3.1).

Proof. To prove that U_2 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0, 1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0, 1)$ is a neutral element.

Note that U_2 differs from U_1 only at points $(1, 0)$ and $(0, 1)$. Since the associativity of U_1 is already known (see Proposition 3.3), we only need to concentrate on the vertices of the unit square.

Since we examine the equation

$$U_2(x, U_2(y, z)) = U_2(U_2(x, y), z), \tag{3.6}$$

we need to take the following possibilities into consideration:

1. For $x = 0$ or $z = 0$ the two sides of the associativity equation in (3.6) are trivially 0.
2. For $x = 1$ and $U_2(y, z) = 0$ by using the strict monotonicity of T we obtain that either $y = 0$ or $z = 0$. This obviously means that the two sides of the associativity equation in (3.6) are equally 0. The proof is similar for $z = 1$ and $U_2(x, y) = 0$.

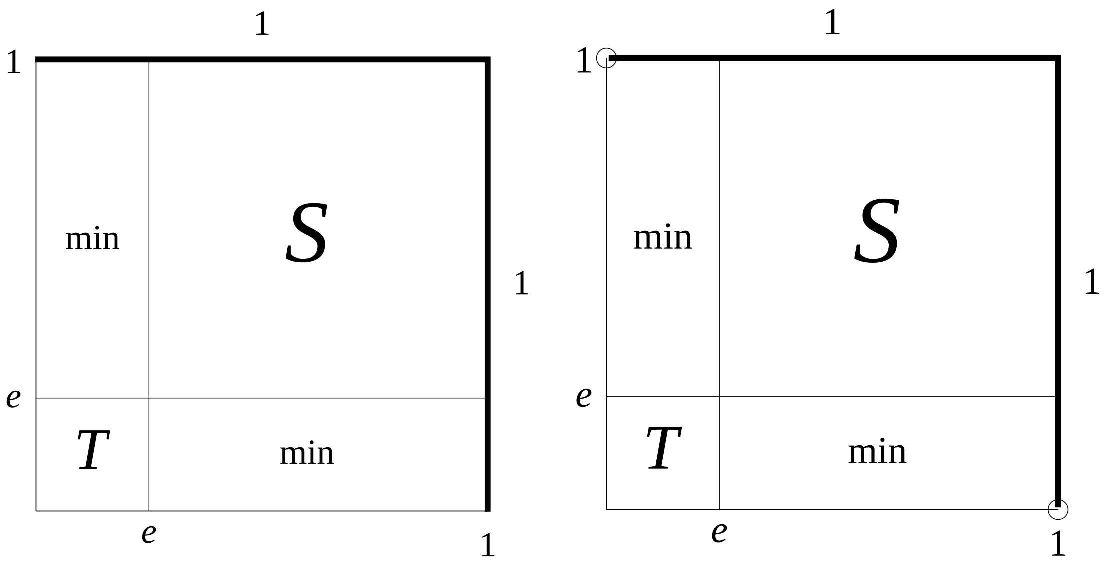
□

Remark 3.7. Note that the strict property cannot be omitted in the Proposition 3.6 (i.e. the statement does not hold for arbitrary t-norms and t-conorms). For a counterexample let us choose $T_{\mathbf{L}}$, $S_{\mathbf{P}}$, $e = 0.3$, $x = 1$, $y = 0.1$, and $z = 0.1$. In this case $U_2(1, U_2(0.1, 0.1)) = 0$, while $U_2(U_2(1, 0.1), 0.1) = 1$.

Proposition 3.8. *(See also Theorem 5 in Li et al., [52].) U_2 in (3.5) is a uninorm if and only if $T(x, y)$ is a positive t-norm, and $S(x, y)$ is dual to a positive t-norm.*

Proof. This condition is sufficient, since in this case the proof is similar to that of Proposition 3.6.

Now I show that it is also necessary. From the proof of Proposition 3.5 the necessity of the second condition is trivial. We only need to show that if $U_2(x, y) = 0$ does not imply $x = 0$ or $y = 0$, then the associativity does not hold. Let us assume indirectly that there exist $y_0, z_0 \neq 0$, for which $U_2(y_0, z_0) = 0$. Obviously, $y_0, z_0 < e$. For $x = 1$ the left hand side of the associativity equation in (3.6) is trivially 0, while the left hand side is 1, which is a contradiction. □



(a) U_1 (b) U_2

FIGURE 3.1: U_1 and U_2

Proposition 3.9. *(See also Li et al., [52].) Let T be a strict t-norm, S be a strict t-conorm and $e \in]0,1[$. The function*

$$U_3(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 0 \text{ or } y = 0; \\ max(x,y), & otherwise. \end{cases} \tag{3.7}$$

is a uninorm with neutral element e (see Figure 3.2).

Proof. To prove that U_3 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0,1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0,1)$ is a neutral element.

Note that U_3 differs from U_{max} only at points where either $x = 0$ or $y = 0$. Since the associativity of $U_{max}(x,y)$ is already known from [40], we only need to concentrate on the border lines where at least one of the variables of U_3 is 0.

To examine the associative equation

$$U_3(x, U_3(y,z)) = U_3(U_3(x,y), z), \tag{3.8}$$

we need to take the following possibilities into consideration:

1. If $x = 0$ or $z = 0$, then $U_3(x, U_3(y, z)) = 0 = U_3(U_3(x, y), z)$.
2. If $U_3(y, z) = 0$ or $U_3(x, y) = 1$, then by using the strict monotonicity of $S(x, y)$, we get $x = 0$ or $y = 0$. Thus $U_3(x, U_3(y, z)) = 0 = U_3(U_3(x, y), z)$.

□

Remark 3.10. Note that the strict property of T cannot be omitted in Proposition 3.9 (i.e. the statement does not hold for arbitrary t-norms). For a counterexample let us choose $T_{\mathbf{L}}$, $S_{\mathbf{P}}$, $e = 0.3$, $x = 0.1$, $y = 0.1$, and $z = 0.8$. In this case $U_3(0.1, U_3(0.1, 0.8)) = 0.8$, while $U_3(U_3(0.1, 0.1), 0.8) = 0$.

Proposition 3.11. *(See also Theorem 4 in Li et al., [52].) U_3 in (3.7) is a uninorm if and only if T is a positive t-norm.*

Proof. The condition is sufficient, since in this case the proof is similar to that of Proposition 3.9.

Now I show that it is also necessary.

Let us assume indirectly that there exist $x_0, y_0 \neq 0$, for which $U_3(x_0, y_0) = 0$. Obviously, $x_0, y_0 < e$. Let $z_0 \neq 0$ so that $U_3(y_0, z_0) \neq 0$. (It is easy to see that such z_0 always exists, since we can always chose $z_0 > e$.) In this case the right hand side of the associativity equation in (3.8) is trivially 0, while the left hand side is z_0, which is a contradiction. □

Remark 3.12. Note that Proposition 3.11 is dual to Proposition 3.5.

Proposition 3.13. *(See also Li et al., [52].) Let T be a strict t-norm, S be a strict t-conorm and $e \in]0, 1[$. The function*

$$U_4(x, y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x, y) \in [0, e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x, y) \in [e, 1]^2; \\ 0, & x = 0, y \neq 1 \text{ or } x \neq 1, y = 0; \\ max(x, y), & otherwise. \end{cases} \tag{3.9}$$

is a uninorm with neutral element e (see figure 3.2).

Proof. To prove that U_4 is a uninorm, we have to show that it is associative, commutative and that it has a neutral element $e \in (0, 1)$. Commutativity is obvious. From the properties of t-norms and t-conorms it follows immediately that $e \in (0, 1)$ is a neutral element.

Note that U_4 differs from U_3 only at points $(1,0)$ and $(0,1)$. Since the associativity of U_3 is already known (see Proposition 3.9), we only need to concentrate on the vertices of the unit square.

Since we examine the equation

$$U_4(x, U_4(y,z)) = U_4(U_4(x,y), z), \tag{3.10}$$

we need to take the following possibilities into consideration:

1. For $x = 1$ or $z = 1$ the two sides of the associativity equation in (3.6) are trivially 1.

2. For $x = 0$ and $U_4(y,z) = 1$ by using the strict monotonicity of $S(x,y)$ we obtain that either $y = 1$ or $z = 1$. This obviously means that the two sides of the associativity equation in (3.6) are equally 1. The proof is similar for $z = 0$ and $U_4(x,y) = 1$.

□

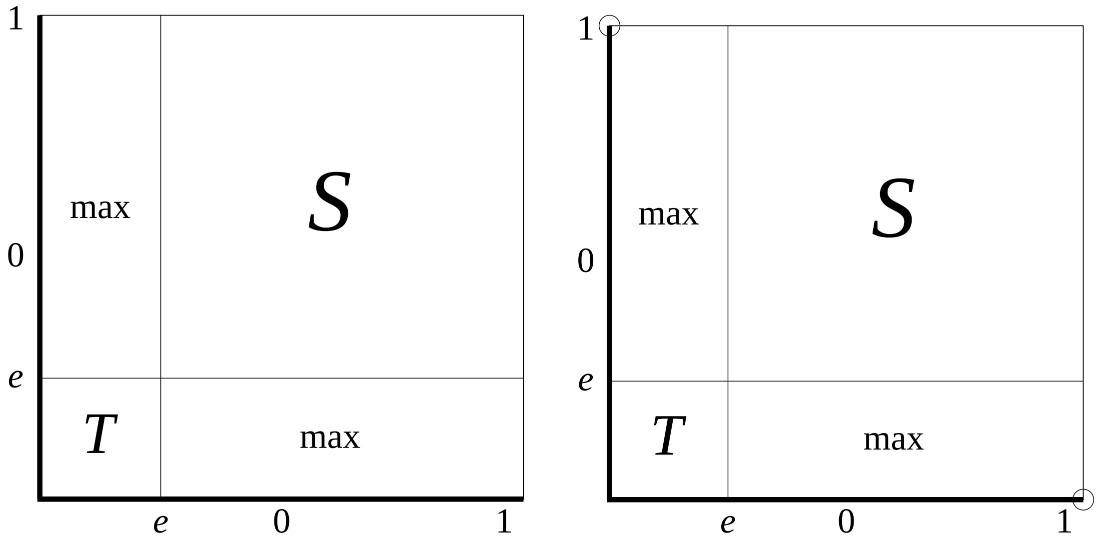
Remark 3.14. Note that the strict property cannot be omitted in the Proposition 3.13 (i.e. the statement does not hold for arbitrary t-norms and t-conorms). For a counterexample let us choose $T_{\mathbf{P}}$, $S_{\mathbf{L}}$, $e = 0.3$, $x = 0$, $y = 0.8$, and $z = 0.9$ In this case $U_4(0, U_4(0.8, 0.9)) = 0$, while $U_4(U_4(0, 0.8), 0.9) = 1$.

Proposition 3.15. *(See also Theorem 5 in Li et al., [52].) U_4 in (3.9) is a uninorm if and only if T is a positive t-norm, and S is dual to a positive t-norm.*

Proof. The condition is sufficient, since in this case the proof is similar to that of Proposition 3.13.

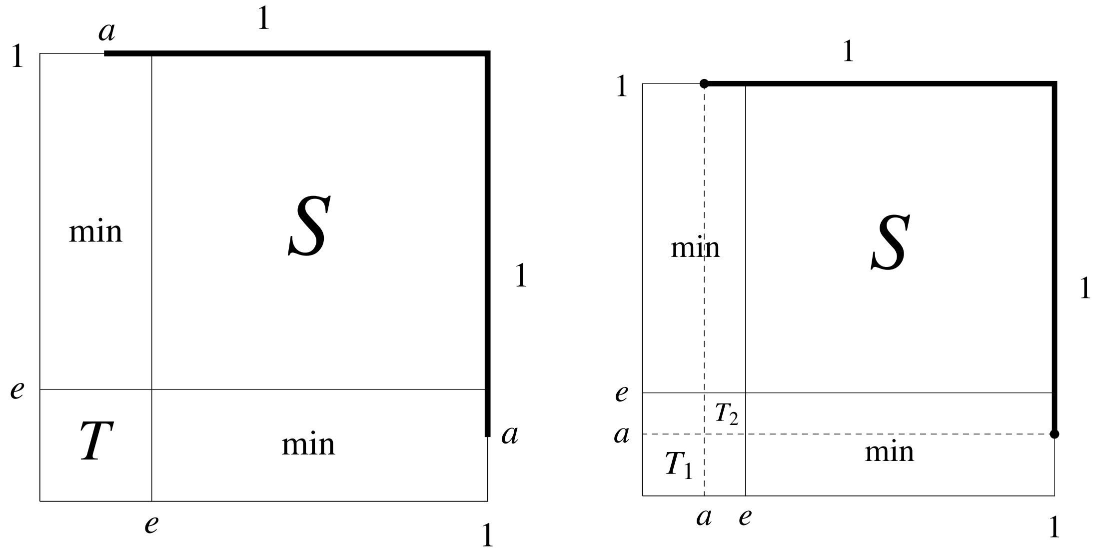
Now I show that it is also necessary. From the proof of Proposition 3.11 the necessity of the first condition is trivial. We only need to show that if $U_4(x,y) = 1$ does not imply $x = 1$ or $y = 1$, then the associativity does not hold. Let us assume indirectly that there exists $y_0, z_0 \neq 1$, for which $U_4(y_0, z_0) = 1$. Obviously, $y_0, z_0 > e$. For $x = 0$ the left hand side of the associativity equation in (3.10) is trivially 1, while the left hand side is 0, which is a contradiction. □

Remark 3.16. Note that Proposition 3.15 is dual to Proposition 3.8.



(a) U_3 (b) U_4

FIGURE 3.2: U_3 and U_4



(a) (b)

FIGURE 3.3: U_5

Let us now consider a function U_5 such that

$$U_5(x,y) = \begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1 \textit{ and } y \geq a \textit{ or } y = 1 \textit{ and } x \geq a; \\ \min(x,y), & \textit{otherwise}. \end{cases} \tag{3.11}$$

where T is a t-norm, S is a t-conorm, $e \in]0,1[$, $a \in]0,e[$ (see Figure 3.3).

Let us consider the conditions under which U_5 can be a uninorm. Suppose that U_5 is a uninorm with neutral element e.

Proposition 3.17. *If U_5 is a uninorm with neutral element e, then $U_5(a,a) = a$.*

Proof. From the conjunctive property of t-norms it follows immediately that $U_5(a,a) \leq a$. Suppose $U_5(a,a) < a$. Then by the definition of U_5, $U_5(1, U_5(a,a)) < 1$. On the other hand, by assiociativity, $U_5(U_5(1,a),a) = U_5(1,a) = 1$, a contradiction. □

Corollary 3.18. *If U_5 is a uninorm with neutral value e, then T is an ordinal sum (see Figure 3.3) of two t-norms, T_1 and T_2, i.e.*

$$T(x,y) = \begin{cases} a \cdot T_1\left(\frac{x}{a}, \frac{y}{a}\right) & \textit{if } (x,y) \in [0,a]^2 \\ a + (e-a) \cdot T_2\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right) & \textit{if } (x,y) \in [a,e]^2 \\ min(x,y) & \textit{otherwise} \end{cases} \tag{3.12}$$

Corollary 3.19. *U_5' and U_5'' defined below are also uninorms (see Figure 3.4):*

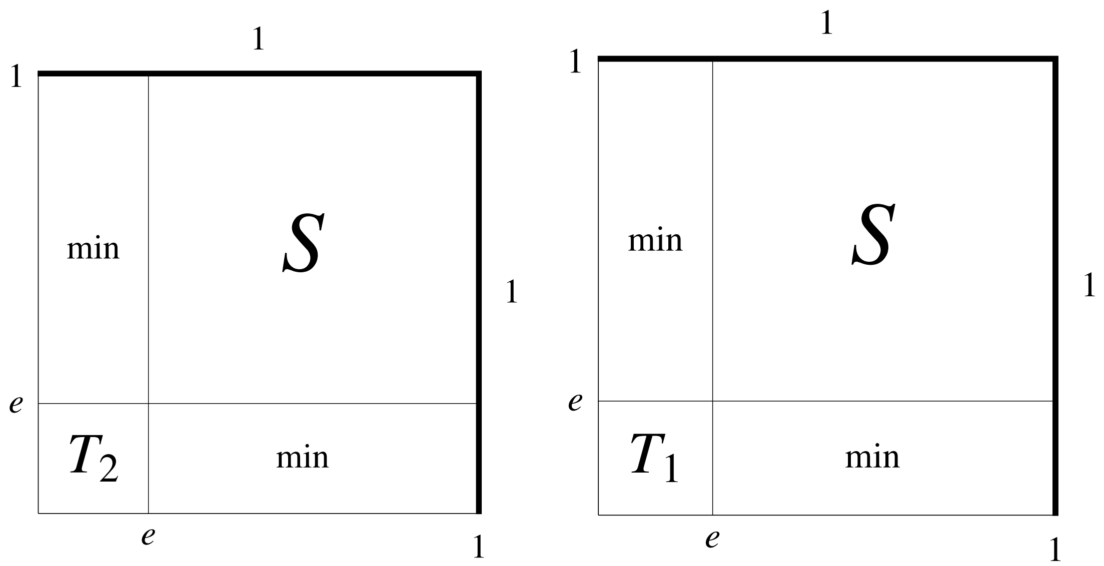
$$U_5'(x,y) = \begin{cases} eT_2\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1 \textit{ or } y = 1; \\ min(x,y), & \textit{otherwise.} \end{cases} \tag{3.13}$$

$$U_5''(x,y) = \begin{cases} eT_1\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 1, & x = 1 \textit{ or } y = 1; \\ min(x,y), & \textit{otherwise.} \end{cases} \tag{3.14}$$

Corollary 3.20. *From Proposition 3.3 and Corollary 3.19 it follows immediately, that if U_5 is a uninorm, then S must be dual to a positive t-norm.*

Proposition 3.21. *U_5 is a uninorm if and only if S is dual to a positive t-norm.*

Proof. The necessity of this condition is the statement of Corollary 3.20. Now I prove that is is also sufficient. Note that U_5 differs from U_{min} only at points (x,y), where $a \leq x < e$ and $y = 1$, or $a \leq y < e$ and $x = 1$. Since the associativity of U_{min} is already known, we only need to concentrate on these regions.



(a) U_5'

(b) U_5''

FIGURE 3.4: U_5' and U_5''

Since we examine the associativity equation

$$U_5(x, U_5(y, z)) = U_5(U_5(x, y), z), \tag{3.15}$$

we need to take the following possibilities into consideration:

1. $a \leq x < e$ and $U(y, z) = 1$. From $U(y, z) = 1$ by the dual-positivity of S we get

   (a) $y = 1$ and $z \geq a$, or

   (b) $y \geq a$ and $z = 1$.

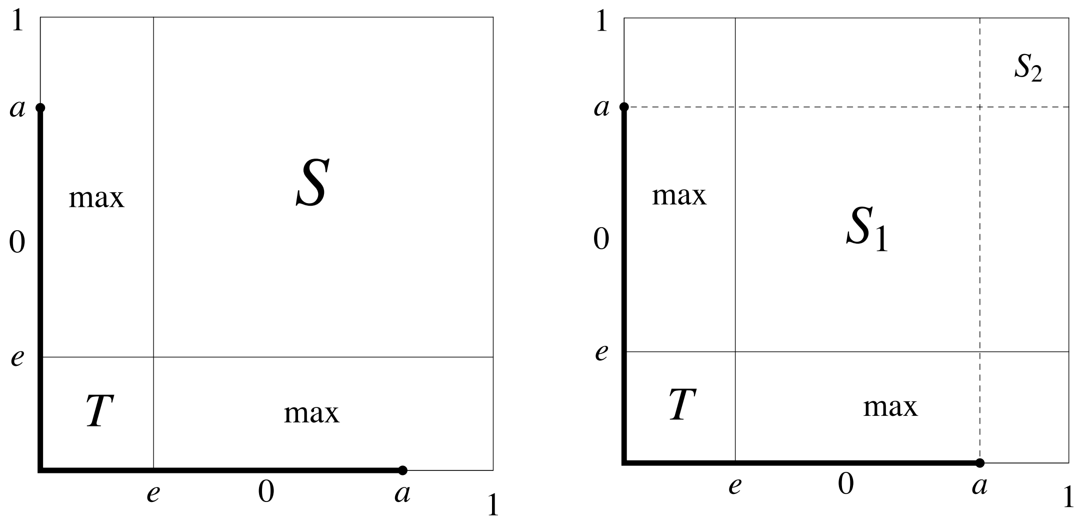
   In case 1a both sides of the associative equation in (3.15) are trivially 1.

   In case 1b the left hand side of (3.15) is 1. In this region

$$U_5(x, y) = \begin{cases} \min(x, y), & y > e, y \neq 1; \\ a + (e - a) \cdot T_2\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right) & a \leq y \leq e; \\ 1, & y = 1, \end{cases}$$

   which means that $U_5(x, y) \geq a$, and therefore the right hand side of (3.15) is also 1.

2. $x = 1$ and $a \leq U_5(y, z) < e$. From the second condition it follows immediately that $a \leq y \leq e$ and $a \leq z \leq e$ hold and therefore both sides of the associativity equation in (3.15) is 1.



(a) (b)

FIGURE 3.5: U_6

3. $z = 1$ and $a \leq U_5(x, y) < e$. The proof is similar to case 2.

4. $U_5(x, y) = 1$ and $a \leq z < e$. The proof is similar to case 1.

$\square$

Now let us define a function U_6 the following way.

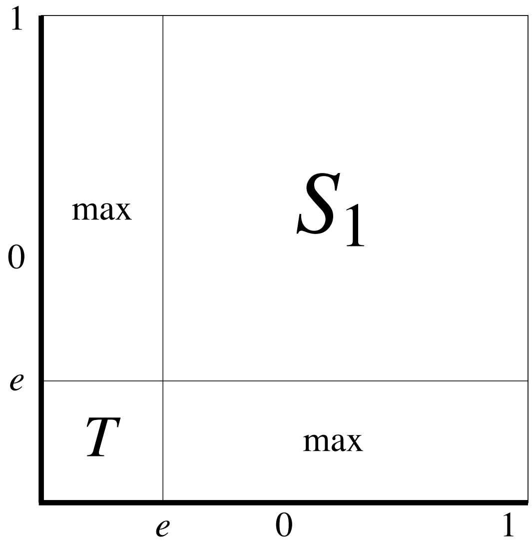
$$U_6(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 0 \textit{ and } y \leq a \textit{ or } y = 0 \textit{ and } x \leq a; \\ \max(x,y), & \textit{otherwise.} \end{cases} \tag{3.16}$$

where T is a t-norm, S is a t-conorm, $e \in]0, 1[$, $a \in]e, 1[$ (see Figure 3.5).

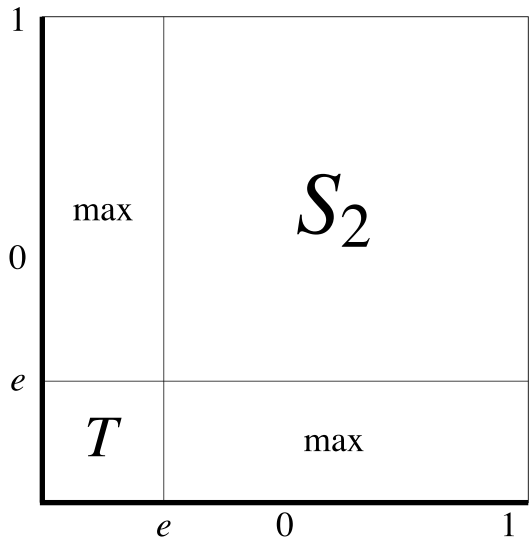
I consider the conditions under which U_6 can be a uninorm. Suppose that U_6 is a uninorm with neutral element e.

Proposition 3.22. *If U_6 is a uninorm with neutral element e, then $U_6(a, a) = a$.*

Proof. From the disjunctive property of t-conorms it follows immediately that $U_5(a, a) \geq a$. Suppose $U_6(a, a) > a$. Then by definition, $U_6(0, U_6(a, a)) = a$, on the other hand by associativity $U_6(0, U_6(a, a)) = U_6(U_6(0, a), a)) = 0$, a contradiction. $\square$



(a) U_6'



(b) U_6''

FIGURE 3.6: U_6' and U_6''

Corollary 3.23. *If U_6 is a uninorm with neutral value e, then S is an ordinal sum of two t-conorms, S_1 and S_2 (see Figure 3.5).*

Corollary 3.24. *U_6' and U_6'' (see Figure 3.6) are also uninorms.*

$$U_6'(x,y) = \begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S_1\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 1 \text{ or } y = 1; \\ max(x,y), & otherwise. \end{cases} \tag{3.17}$$

$$U_6''(x,y) = \begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right), & (x,y) \in [0,e]^2; \\ e + (1-e)S_2\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right), & (x,y) \in [e,1]^2; \\ 0, & x = 1 \text{ or } y = 1; \\ max(x,y), & otherwise. \end{cases} \tag{3.18}$$

Corollary 3.25. *From Proposition 3.3 and Corollary 3.24 it follows immediately, that if U_6 is a uninorm, then T must be a positive t-norm.*

Proposition 3.26. *U_6 is a uninorm if and only if T is a positive t-norm.*

Proof. The necessity of this condition is the statement of Corollary 3.25. Now I prove that is is also sufficient. Note that U_6 differs from U_{max} only at points (x,y), where

$e < x \leq a$ and $y = 0$, or $e < y \leq a$ and $x = 0$. Since the associativity of U_{max} is already known, we only need to concentrate on these regions.

Since we examine the associativity equation

$$U_6(x, U_6(y, z)) = U_6(U_6(x, y), z), \tag{3.19}$$

we need to take the following possibilities into consideration:

1. $e < x \leq a$ and $U(y, z) = 0$. From $U(y, z) = 0$ by the positivity of T we get

   (a) $y = 0$ and $z \leq a$, or

   (b) $y \leq a$ and $z = 0$.

   In case 1a both sides of the associative equation in (3.19) are trivially 0.

   In case 1b the left hand side of (3.19) is 0. In this region

$$U_6(x, y) = \begin{cases} e + (1-e) \cdot S\left(\frac{x-a}{e-a}, \frac{y-a}{e-a}\right) & e \leq y \leq a; \\ \max(x, y), & 0 < y < e, y \neq 1; \\ 0, & y = 0, \end{cases}$$

   which means that $U_6(x, y) \leq a$, and therefore the right hand side of (3.19) is also 0.

2. $x = 0$ and $e < U_6(y, z) \leq a$. From the second condition it follows immediately that $e \leq y \leq a$ and $e \leq z \leq a$ hold and therefore both sides of the associativity equation in (3.15) is 0.

3. $z = 0$ and $e < U_6(x, y) \leq a$. The proof is similar to case 2.

4. $U_6(x, y) = 0$ and $e < z \leq a$. The proof is similar to case 1.

□

3.3 Overview

In this section, some new construction methods of uninorms with fixed values along the borders were discussed.

These uninorms differ from U_{min} or U_{max} only at points along their boarder lines.

Sufficient and necessary conditions were presented. Our results show that U_1 in (3.3) is a uninorm if and only if S is dual to a positive t-norm. U_2 in (3.5) is a uninorm if and only if $T(x, y)$ is a positive t-norm, and $S(x, y)$ is dual to a positive t-norm. U_3 in (3.7) is a uninorm if and only if T is a positive t-norm. U_4 in (3.9) is a uninorm if and only if T is a positive t-norm, and S is dual to a positive t-norm. U_5 is a uninorm if and only if S is dual to a positive t-norm. U_6 is a uninorm if and only if T is a positive t-norm.

Thesis 1.2. New construction methods of uninorms with fixed values along the borders are presented. Sufficient and necessary conditions are presented.

Chapter 4

The General Nilpotent Operator System

4.1 Motivation and scope

One of the most significant problems of fuzzy set theory is the proper choice of set-theoretic operations [68, 77]. Triangular norms and conorms have thoroughly been examined in the literature [36, 38, 43, 51]. The most well-characterized class of t-norms are the so-called representable t-norms, derived from the solution of the associative functional equation [1]. Representable t-norms and t-conorms are often used as conjunctions and disjunctions in logical structures [42], [62]. Henceforth I refer to representable t-norms as conjunctions ($c(x, y)$) and representable t-conorms as disjunctions ($d(x, y)$). It is important to note that all strict t-norms are isomorphic to the product torm, while all nilpotent t-norms are isomorphic to Łukasiewicz t-norm [43]. Łukasiewicz fuzzy logic [45, 54, 63, 65] is the logic where the conjunction is the Łukasiewicz t-norm and the disjunction is the Łukasiewicz t-conorm.

The class of non-strict t-norms has preferable properties which make them more usable in building up logical structures. Among these properties are the fulfillment of the law of contradiction and the excluded middle, the continuity of the implication or the coincidence of the residual and the S-implication [33, 74]. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm [43], the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic.

In this section it is shown that a consistent logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic. Of course, this lack of isomorphy is not the result of introducing a new operator family, it simply means that the system itself is built up in a significantly different way using more than one generator functions.

This section is organized as follows. First, a characterization of negation operators is given in Section 4.2, as negations will have an important role to play in Section 4.3. After considering the class of connective systems generated by nilpotent operators, their structural properties are examined in Section 4.3. Examples for bounded systems, i.e. consistent nilpotent systems which are not isomorphic to Łukasiewicz-logic are shown. Necessary and sufficient conditions are given for these systems to satisfy the De Morgan law, classification property and consistency. A wide range of examples for consistent and non-consistent bounded systems can be found in Section 4.4.

The results of this chapter can be found in Dombi and Csiszár, [27].

4.2 Characterization of strict negation operators

The main goal of this section is to present a representation of strict negations with a wide range of examples, since negations will have a very important role to play in the next section.

First let us see some further examples for negation operators. Hamacher proved in [46] that the only negation having polynomial form is $1-x$, the so-called standard negation, introduced by Zadeh in [82]. He also proved that if an involutive negation belongs to the class of rational polynomials, then it has the following form:

$$n_\lambda(x) = \frac{1-x}{1+\lambda x}, \quad \text{where} \quad \lambda > -1. \tag{4.1}$$

Sugeno [71] had the same result from the concept of fuzzy measures and integrals.

In the literature, generally the standard negation $1-x$ or infrequently $\frac{1-x}{1+x}$ (4.1) for λ =1) are used. Here I make suggestions about using different types of negations as well. The negation operators can be characterized by their neutral values. In [25] (see also [30]) Dombi introduced the following negation formula by expressing $n_\lambda(x)$ with the help of its neutral element ν_*:

$$n_{\nu_*}(x) = \frac{1}{1+(\frac{1-\nu_*}{\nu_*})^2\frac{x}{1-x}}. \tag{4.2}$$

Note that if $\nu_* \to 0$, then $\lim n_{\nu_*}(x) = n_0(x)$, if $\nu_* \to 1$, then $\lim n_{\nu_*}(x) = n_1(x)$ and for $\nu_* = \frac{1}{2}$ we get the standard negation.

Yager [78] introduced

$$n(x) = (1 - x^{\alpha})^{1/\alpha}, \quad \alpha > 0. \tag{4.3}$$

Both this type of negation operator and the above-mentioned n_λ reduce to the standard negation when $\alpha = 1$ and $\lambda = 0$ respectively.

It is easy to see that the neutral value of the negation operator in (4.3) is $2^{-\frac{1}{\alpha}}$. If we write this negation operator by using its neutral value as a parameter, we get $n(x) = \left(1 - x^{-\frac{1}{\log_2 \nu_*}}\right)^{-\log_2 \nu_*}$.

Note that the representation in Proposition 1.4 is not unique. It is not always easy to find a generator function. The following propositions state that there can be infinitely many generator functions for a negation operator.

Proposition 4.1. *Let* $\nu_* \in (0,1)$, $f : [0,1] \to [0,1]$

$$f(x) = \begin{cases} \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*} \cdot \frac{1-x}{x}\right)^{\alpha}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is a generator function of the negation n_{ν_*} *(see* (4.2)*) for any* $\alpha \neq 0$.

Proof. It can easily be seen that $f^{-1}(x) = \frac{1}{1+\frac{\nu_*}{1-\nu_*}\left(\frac{1-x}{x}\right)^{\frac{1}{\alpha}}}$, and $1 - f(x) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*} \cdot \frac{1-x}{x}\right)^{-\alpha}}$, hence $f^{-1}(1 - f(x)) = \frac{1}{1+\left(\frac{1-\nu_*}{\nu_*}\right)^2 \frac{x}{1-x}} = n_{\nu_*}(x)$.

□

Remark 4.2. Note that in Proposition 4.1, if f is a generator function of n, then f^{-1} also generates n.

Proposition 4.3. *In Proposition 1.3 (Trillas) the generator function can also be decreasing.*

Proof. I shall prove that if f_n is a generator function of n, then $g_n(x) = 1 - f_n(x)$ is also a generator function of n. If f_n is the generator function of n, then $n(x) = f^{-1}(1 - f_n(x))$. If $g_n(x) = 1 - f_n(x)$ then $g_n^{-1}(x) = f_n^{-1}(1 - x)$. With this generator function the negation has the following form: $g^{-1}(1 - g(x)) =$
$= g^{-1}(1 - (1 - f_n(x))) = g^{-1}(f_n(x)) = f_n^{-1}(1 - f_n(x))$. Since f_n is increasing, g_n is decreasing. □

For the neutral element ν_*, using the representation theorem, we get $\nu_* = f^{-1}(1 - f(\nu_*))$, so $\nu_* = f^{-1}\left(\frac{1}{2}\right)$.

For the generator function $g(x) = \frac{a^x - 1}{a-1}$, where $a > 0, a \neq 1$, we get

$$n(x) = \log_a \left(a + 1 - a^x\right). \tag{4.4}$$

If we choose the inverse function $g^{-1}(x) = \log_a (x(a-1)+1)$ for the generator function, we obtain

$$n(x) = \frac{1-x}{1+x(a-1)}, \tag{4.5}$$

which was mentioned above.

In this section three basic families of strict negations generated by rational, power and exponential functions were considered. (See also Tables A.1, A.3 and A.4.)

4.3 Nilpotent connective systems

Next, instead of operators in themselves, connective systems are considered.

Definition 4.4. The triple (c, d, n), where c is a t-norm, d is a t-conorm and n is a strong negation, is called a connective system.

Definition 4.5. A connective system is nilpotent if the conjunction c is a nilpotent t-norm, and the disjunction d is a nilpotent t-conorm.

Definition 4.6. Two connective systems (c_1, d_1, n_1) and (c_2, d_2, n_2) are isomorphic if there exists a bijection $\phi : [0,1] \to [0,1]$ such that

$$\phi^{-1}\left(c_1\left(\phi(x), \phi(y)\right)\right) = c_2(x,y)$$

$$\phi^{-1}\left(d_1\left(\phi(x), \phi(y)\right)\right) = d_2(x,y)$$

$$\phi^{-1}\left(n_1\left(\phi(x)\right)\right) = n_2(x).$$

In the nilpotent case, the generator functions of the disjunction and the conjunction being determined up to a multiplicative constant can be normalized the following way:

$$f_c(x) := \frac{t(x)}{t(0)}, \qquad f_d(x) := \frac{s(x)}{s(1)}.$$

Remark 4.7. Thus, the normalized generator functions are uniquely defined.

I will use normalized generator functions for conjunctions and disjunctions well. This means that the normalized generator functions of conjunctions, disjunctions and negations are

$$f_c, f_d, f_n : [0,1] \to [0,1].$$

I will suppose that f_c is continuous and strictly decreasing, f_d is continuous and strictly increasing and f_n is continuous and strictly monotone.

Note that by using Proposition 4.3, there are two special negations generated by the normalized additive generators of the conjunction and the disjunction.

Definition 4.8. The negations n_c and n_d generated by f_c and f_d respectively,

$$n_c(x) = f_c^{-1}(1 - f_c(x))$$

and

$$n_d(x) = f_d^{-1}(1 - f_d(x))$$

are called natural negations of c and d respectively.

This means that for a connective system with normalized generator functions f_c, f_d and f_n we can associate three negations by (1.1), n_c, n_d and n.

Proposition 4.9. *With the help of the cutting operator (see Definition 1.9), we can write the conjunction and disjunction in the following form, where f_c and f_d are decreasing and increasing normalized generator functions respectively.*

$$c(x, y) = f_c^{-1}[f_c(x) + f_c(y)], \tag{4.6}$$

$$d(x, y) = f_d^{-1}[f_d(x) + f_d(y)]. \tag{4.7}$$

Proof. From (1.2) we know that

$$c(x, y) = f_c^{-1}\left(\min(f_c(x) + f_c(y), f_c(0)\right) = f_c^{-1}\left(\min(f_c(x) + f_c(y), 1\right) = f_c^{-1}[f_c(x)+f_c(y)],$$

and similarly, from (1.3)

$$d(x, y) = f_d^{-1}\left(\min(f_d(x) + f_d(y), f_d(0)\right) = f_d^{-1}\left(\min(f_d(x) + f_d(y), 1\right) = f_d^{-1}[f_d(x)+f_d(y)].$$

□

Remark 4.10. Note that in Proposition 4.9 it is necessary to use normalized generator functions as the following example shows. This fact supports the use of normalized functions.

Example 4.1. *Let $f_c(x) = 2 - 2x$.*

$$c\left(\frac{1}{2}, \frac{1}{2}\right) = f_c^{-1}\left(min\left(f_c(x) + f_c(y), f_c(0)\right)\right) = f_c^{-1}(2) = 0,$$

while

$$f_c^{-1}\left[f_c\left(\frac{1}{2}\right)+f_c\left(\frac{1}{2}\right)\right]=f_c^{-1}\left[2-1+2-1\right]=f_c^{-1}[2]=f_c^{-1}(1)=\frac{1}{2}.$$

Remark 4.11. Note that using the cutting function defined above we can omit applying the min and max operators. In the literature, the use of the pseudo-inverse was replaced by the forms (1.2) and (1.3), which is now replaced by (4.6) and (4.7).

Definition 4.12. A connective system is called Łukasiewicz system if it is isomorphic to $([x+y-1],[x+y],1-x)$, i.e. if there exists a bijection $\phi:[0,1]\to[0,1]$ such that the connective system has the form $(\phi^{-1}[\phi(x)+\phi(y)-1],\phi^{-1}[\phi(x)+\phi(y)],\phi^{-1}[1-\phi(x)])\quad for\ \forall x,y\in[0,1]$.

Proposition 4.13. *For nilpotent t-norms and t-conorms Definition 4.8 is equivalent to the following definition (also denoted by N_T and N_S, see Klement et al., [51], p. 232 and Baczyński and Jayaram, [4], Definition 2.3.1.):*

$$n_c(x)=N_T(x)=\ \ sup\ \{y\in[0,1]\mid c(x,y)=0\},\quad x\in[0,1],$$

$$n_d(x)=N_S(x)=\ \ inf\ \{y\in[0,1]\mid d(x,y)=1\},\quad x\in[0,1].$$

Proof. For the conjunction, $c(x,y)=f_c^{-1}[f_c(x)+f_c(y)]=0$ iff $f_c(x)+f_c(y)\geq 1$, from which $y\leq f_c^{-1}\left(1-f_c(x)\right)=n_c(x)$. For $y=n_c(x)$, $c(x,n_c(x))=0$ is trivial. The proof is similar for the disjunction as well. □

4.3.1 Structural properties of connective systems

Definition 4.14. Classification property means that the law of contradiction holds, i.e.

$$c(x,n(x))=0,\qquad \forall x,y\in[0,1], \tag{4.8}$$

and the excluded third principle holds as well, i.e.

$$d(x,n(x))=1,\qquad \forall x,y\in[0,1]. \tag{4.9}$$

Definition 4.15. The **De Morgan identity** means that

$$c(n(x),n(y))=n(d(x,y)) \tag{4.10}$$

or

$$d(n(x),n(y))=n(c(x,y)). \tag{4.11}$$

Remark 4.16. These two forms of the De Morgan law are equivalent if the negation is involutive. The first De Morgan law holds with a strict negation n if and only if the second holds with n^{-1} (see Fodor and Roubens, [38], p. 18)

Definition 4.17. A connective system is said to be **consistent** if the *classification property* (Definition 4.14) and the *De Morgan identity* (Definition 4.15) hold.

4.3.1.1 Classification property

Now I will examine the conditions that the connectives and their normalized generator functions in a connective system must satisfy if we want the classification property to hold.

Proposition 4.18. *(See also Fodor and Roubens, [38], 1.5.4. and 1.5.5., and Baczyński and Jayaram, [4], 2.3.2.) In a connective system (c, d, n) the classification property holds iff*

$$n_d(x) \leq n(x) \leq n_c(x), \quad for \quad \forall x \in [0, 1]$$

where n_c and n_d are the natural negations of c and d, respectively.

Proof. From the excluded third principle, we have $d(x, n(x)) = 1$. Using the normalized generator function, $f_d^{-1}[f_d(x) + f_d(n(x))] = 1$. It means that $f_d(x) + f_d(n(x)) \geq 1$, from which $f_d(n(x)) \geq 1 - f_d(x)$.

f_d and its inverse f_d^{-1} are strictly increasing, thus we get the left hand side of the inequality:

$$n(x) \geq f_d^{-1}(1 - f_d(x)) = n_d(x).$$

Similarly, we get the right hand side from the law of contradiction $c(x, n(x)) = 0$. Using the normalized generator function we get $f_c^{-1}[f_c(x) + f_c(n(x))] = 0$. From the definition of the cutting function $f_c(x) + f_c(n(x)) \geq 1$, which means that $f_c(n(x)) \geq 1 - f_c(x)$. Since f_c and f_c^{-1} are strictly decreasing,

$$n(x) \leq f_c^{-1}(1 - f_c(x)) = n_c(x),$$

$$n_d(x) \leq n(x) \leq n_c(x).$$

□

Remark 4.19. Generally, in a consistent system only one negation is used in the literature. The logical connectives are usually generated by a single generator function.

$$c(x, y) = f^{-1}\left[f(x) + f(y) - 1\right],$$

$$d(x,y) = f^{-1}\left[f(x) + f(y)\right],$$

$$n(x) = f^{-1}\left(1 - f(x)\right),$$

where $f : [0,1] \to [0,1]$ is a continuous, strictly increasing function.

The question arises immediately, whether the use of more than one negation is possible. This possibility will be considered later in detail (see 4.3.2.1).

Next I give examples for connective systems in which the classification property holds, but which does not fulfil the De Morgan law.

In Section 4.4, an overview of all the examples included in the following part of this section is presented. The examples from the rational family will be considered in detail in 4.3.2.1.

Example 4.2. *Let $f_n(x) := x^2$, $f_c(x) := \sqrt{1-x}$ and $f_d(x) := \sqrt{x}$. This connective system fulfills the classification property but does not fulfill the De Morgan law. (See also Table A.1.)*

Another example can be obtained by using the rational family of normalized generators functions

$$f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}, \quad f_n(0) = 0,$$

$$f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}, \quad f_c(1) = 0,$$

$$f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}, \quad f_d(0) = 0,$$

choosing e.g. $\nu_d = 0.3$, $\nu_c = 0.7$ and $\nu = 0.5$. (See Table A.4.)

The existence of such systems explains why we have to consider the De Morgan law in the following section.

4.3.1.2 The De Morgan law

Now I will examine the conditions that the connectives and their normalized generator functions must satisfy, if we want the connective system to fulfill the De Morgan law. Before stating Proposition 4.23 we need to solve the following functional equation.

Lemma 4.20. *Let $u : [0,1] \longrightarrow [0,1]$ be a continuous, strictly increasing function with $u(0) = 0$ and $u(1) = 1$. The functional equation*

$$[u(x) + u(y)] = u[x + y] \tag{4.12}$$

(where [ ] stands for the cutting operator defined in Definition 1.9) has a unique solution $u(x) = x$.

Proof. • First I prove that $u[0] = 0$. Let us suppose that $u[0] = c$, where $0 \leq c \leq 1$. Then

$$c = u[0 + 0] = [2u(0)],$$

which means $c = [2c]$ i.e. $c = 1$, or $c = 0$, but $c = 1$ contradicts $u(0) = 0$.

- Second, I show that $u[1] = 1$. Similarly, let us suppose that $u[1] = c$, where $0 \leq c \leq 1$. Then $c = u[1 + 1] = [2u(1)]$, which means $c = [2c]$ i.e. $c = 1$, or $c = 0$, but for $c = 0$ we get contradiction.

- Third, I prove that $u\left(\frac{1}{2}\right) = \frac{1}{2}$.

  If $x < \frac{1}{2}$, then $2x < 1$. u is strictly increasing, therefore $u(2x) < 1$ as well. $u[2x] = u(2x) = 2u(x) = [2u(x)]$, because of the continuity of u, $\lim_{x \to \frac{1}{2}} u(2x) = u(1)$, $2 \lim_{x \to \frac{1}{2}} u(x) = 1$, which implies $u\left(\frac{1}{2}\right) = \frac{1}{2}$.

- Similarly, we can prove that $u\left(\frac{1}{2^m}\right) = \frac{1}{2^m}$.

- Next, I prove that $u\left(\frac{3}{4}\right) = \frac{3}{4}$.

  $u\left(\frac{3}{4}\right) = u\left(\frac{1}{2} + \frac{1}{4}\right) = u\left(\frac{1}{2}\right) + u\left(\frac{1}{4}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

- In a similar way, we obtain that for $u\left(\frac{k}{2^m}\right) = \frac{k}{2^m}$.

  Then, for any rational number from $[0, 1]$, we have $u(x) = x$.

- Let r be any arbitrary irrational number from $[0, 1]$. There exists a sequence of rational numbers q_n such that $\forall n : q_n \in [0, 1]$ and $q_n \longrightarrow r$.

Because of the continuity of u we have $u(q_n) \longrightarrow u(r)$, which implies $u(r) = r$. □

Note that the solution of the following general form of the functional equation (4.12) can be found in the papers of Baczynski [6], [7] (Propositions 3.4. and 3.6.).

Proposition 4.21. *Fix real $a, b > 0$. For a function $f : [0, a] \to [0, b]$, the following statements are equivalent.*

1. *f satisfies the functional equation*
   $f(min(x + y, a)) = min(f(x) + f(y), b) \quad \forall x, y \in [0, a]$.

2. *Either $f = b$, or $f = 0$, or*

$$f(x) = \begin{cases} 0 \text{ if } & x = 0 \\ b \text{ if } & 0 < x \leq a \end{cases}$$

or there exists a unique constant $c \in [b/a, \infty)$ *such that*

$$f(x) = min(cx, b), \qquad x \in [0, a].$$

Remark 4.22. Specially, for $a = b = 1$ we get the statement of Lemma 4.20.

Proposition 4.23. *If* f_c *is the normalized generator function of a conjunction in a connective system,* f_d *is a normalized generator function of the disjunction and* n *is a strong negation, then the following statements are equivalent:*

1. *The De Morgan law holds in the connective system. That is,*

$$c(n(x), n(y)) = n(d(x, y)). \tag{4.13}$$

2. *The normalized generator functions of the conjunction, disjunction and negation operator obey the following equations (which are obviously equivalent to each other):*

$$n(x) = f_c^{-1}\left(f_d(x)\right) = f_d^{-1}\left(f_c(x)\right), \tag{4.14}$$

$$f_c(x) = f_d(n(x)) \quad \text{ or equivalently } \quad f_d(x) = f_c(n(x)). \tag{4.15}$$

Proof. (4.15) $\Rightarrow$ (4.13) is obvious.

(4.13) $\Rightarrow$ (4.14): Let us write the De Morgan law using the normalized generator functions.

$$f_c^{-1}[f_c(n(x)) + f_c(n(y))]) = n(f_d^{-1}[f_d(x) + f_d(y)]).$$

Applying $f_c(x)$ to both sides of the equation we obtain

$$[f_c(n(x)) + f_c(n(y))] = f_c(n(f_d^{-1}[f_d(x) + f_d(y)])).$$

Let us substitute $x = f_d^{-1}(x)$. Then we have

$$[f_c(n(f_d^{-1}(x))) + f_c(n(f_d^{-1}(y)))] = f_c(n(f_d^{-1}[f_d(f_d^{-1}(x)) + f_d(f_d^{-1}(y))])).$$

From this, we get the following functional equation:

$$[f_c(n(f_d^{-1}(x))) + f_c(n(f_d^{-1}(y)))] = f_c(n(f_d^{-1}[x + y])).$$

If we use $u(x) := f_c(n(f_d^{-1}(x)))$, then we get the following form of the functional equation:

$$[u(x) + u(y)] = u[x + y].$$

We can readily see that function $u(x)$ satisfies the conditions of Lemma 4.20, i.e. it is a continuous, strictly monotone increasing function with $u(0) = 0$ and $u(1) = 1$. This means that by Lemma 4.20, $u(x) = x$. Hence, $f_c\left(n\left(f_d^{-1}(x)\right)\right) = x$. □

Remark 4.24. Note that in Proposition 4.23 any two of n, f_c, f_d determine the third.

However, note that this remark above does *not* mean that any two of n, f_c, f_d can be chosen arbitrary. If f_c and f_d are given and we want the De Morgan property to hold, we obtain n from ((4.14)). This means that for f_c and f_d the equation in (4.14) has to hold. Hence, in order to get an involutive negation, we must take notice of the appropriate relationship of the normalized generator functions as the following example shows.

Example 4.3. *Let $f_c(x) = 1 - x^\alpha$ and $f_d(x) = x^\beta$, where $\alpha \neq \beta$. Then*

$$f_c^{-1}\left(f_d(x)\right) = \sqrt[\alpha]{1 - x^\beta} \neq \sqrt[\beta]{1 - x^\alpha} = f_d^{-1}\left(f_c(x)\right).$$

Proposition 4.25. *If the De Morgan property holds in a connective system (c, d, n), then*

$$n_c\left(n(x)\right) = n\left(n_d(x)\right) \tag{4.16}$$

and similarly,

$$n_d\left(n(x)\right) = n\left(n_c(x)\right), \tag{4.17}$$

where n_c and n_d are the natural negations.

Proof. Because of the involutive property of n it is enough to prove (4.16).

$$n\left(f_c^{-1}\left(1 - f_c\left(n(x)\right)\right)\right) = f_d^{-1}\left(f_c\left(f_c^{-1}\left(1 - f_c\left(f_c^{-1}\left(f_d(x)\right)\right)\right)\right)\right) = n_d(x).$$

□

Corollary 4.26. *If the De Morgan law holds in a connective system (c, d, n), then*

$$n(x) = n_c(x) \text{ if and only if } n(x) = n_d(x), \tag{4.18}$$

where n_c and n_d are the natural negations.

Remark 4.27. Note that we can readily see that if any two of n, n_d, n_c are equal, then the third is equal to them as well.

Proposition 4.28. *Let h be the transformation for which $h\left(f_c(x)\right) = f_d(x)$ in a connective system in which the De Morgan property holds. Then h is a (strong) negation.*

Proof. By using the involutive property of n, we get

$$f_d^{-1}\left(f_c(x)\right) = f_c^{-1}\left(f_d(x)\right),$$

$$f_d(x) = f_c\left(f_d^{-1}\left(f_c(x)\right)\right),$$

$$f_c(x) = f_d\left(f_c^{-1}\left(f_d(x)\right)\right) = h\left(f_d(x)\right),$$

$$f_c^{-1}(x) = f_d^{-1}\left(h^{-1}(x)\right),$$

$$f_d\left(f_c^{-1}(x)\right) = h^{-1}(x) = h(x).$$

So h is also involutive. It is easy to see that $h(0) = 1, h(1) = 0$ and $h(x) = f_d\left(f_c^{-1}(x)\right)$ is strictly monotone decreasing. □

Now I give examples for consistent and non-consistent connective systems where the De Morgan property holds. For examples from the rational family of normalized generator functions see propositions 4.38 and 4.40.

Example 4.4. *If in a connective system the conjunction, the disjunction and the negation have the following forms*

$$f_n(x) = x, f_c(x) = (1-x)^\alpha, f_d(x) = x^\alpha,$$

then this connective system is consistent (i.e. the De Morgan law and the classification property hold), if and only if $0 < \alpha \leq 1$. *(See also Table A.1.)*

Proof. It is easy to see, that from the Proposition 4.15 formula (4.14) is true for the mentioned normalized generator and negation functions:

$$x^\alpha = (1-(1-x))^\alpha,$$

which means that the De Morgan law holds.
It is easy to see that the classification property holds if and only if

$$x^\alpha + (1-x)^\alpha \geq 1,$$

which is only true if for $0 < \alpha \leq 1$. □

Remark 4.29. Note that the example above shows that there exists a system in which the De Morgan property holds, whereas the classification property does not (for $\alpha > 1$). (See also Table A.1.)

For an example from the rational family of normalized generator functions (see propositions 4.38 and 4.40 and also Table A.4)

$$f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}, \quad f_n(0) = 0,$$

$$f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}, \quad f_d(0) = 0,$$

$$f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}, \quad f_c(1) = 0,$$

we can choose e.g. $\nu = 0.6$, $\nu_c = 0.2$ and $\nu_d = 0.36$.

Example 4.5. *If we express the normalized generator functions in Example 4.4 in terms of the neutral values of the related negations, we get*

$$f_n(x) = x, f_c(x) = (1-x)^{\frac{1}{log_{0.5}(1-\nu_c)}}, f_d(x) = x^{log_{\nu_d}(0.5)}.$$

This system fulfills the De Morgan identity iff $\nu_c + \nu_d = 1$, and is consistent iff $\nu_d \leq \frac{1}{2}$ also holds. (See also Table A.1.)

4.3.2 Consistent connective systems

Now consistent connective systems (in which the De Morgan property and the classification property hold together) are to be considered.

Proposition 4.30.
1. *If the connective system (c, d, n) is consistent, then $f_c(x) + f_d(x) \geq 1$ for any $x \in [0, 1]$, where f_c and f_d are the normalized generator functions of the conjunction c and the disjunction d respectively.*
2. *If $f_c(x) + f_d(x) \geq 1$ for any $x \in [0, 1]$ and the De Morgan law holds, then the connective system (c, d, n) satisfies the classification property as well (which now means that the system is consistent).*

Proof. By Proposition 4.18, the classification property holds if and only if

$$f_d^{-1}(1 - f_d(x)) = n_d(x) \leq n(x) \leq n_c(x) = f_c^{-1}(1 - f_c(x))$$

and by Proposition 4.23, the De Morgan identity holds if and only if

$$n(x) = f_d^{-1}(f_c(x)) = f_c^{-1}(f_d(x)).$$

From the right hand side of the inequality we get

$$f_c^{-1}\left(f_d(x)\right) \leq f_c^{-1}\left(1 - f_c(x)\right),$$

so

$$f_c(x) + f_d(x) \geq 1.$$

Similarly, we get the same from the left hand side of the inequality.

$\square$

Remark 4.31. Note that as Example 4.2 shows, $f_c(x) + f_d(x) \geq 1$ does not imply the De Morgan law, even if the classification property holds.

Moreover, $f_c(x) + f_d(x) \geq 1$ without the De Morgan law does not imply the classification property either (for a counterexample we can chose $f_n = x^2$ and $\alpha = 0.7$ in Example 4.4).

Next, examples for consistent systems are presented.

Example 4.6. *If in a connective system the generator function of the conjunction, the disjunction and the negation have the following forms*

$$f_c(x) = 1 - x^\alpha, f_d(x) = x^\alpha, f_n(x) = x^\alpha,$$

where $\alpha > 0$*, then the De Morgan law and the classification property hold for every* α*. (See also Table A.1.)*

Example 4.7. *More generally, the connective system with generator functions*

$$f_c(x) = (1 - x^\alpha)^{\frac{\beta}{\alpha}}, f_d(x) = x^\beta, f_n(x) = x^\alpha,$$

where $\alpha, \beta > 0$ *is consistent if and only if* $\beta \leq \alpha$*. (See also Table A.1.)*

Note that Example 4.7 reduces to Example 4.4 if $\alpha = 1$ and $0 < \beta \leq 1$ and to Example 4.6 if $\alpha = \beta$.

Proposition 4.32. *In a connective system the following equations are equivalent:*

$$f_c(x) + f_d(x) = 1 \tag{4.19}$$

$$n_c(x) = n_d(x), \tag{4.20}$$

where f_c, f_d *are the normalized generator functions of the conjunction and the disjunction and* n_c, n_d *are the natural negations.*

Proof. From $f_d(x) = 1 - f_c(x)$,

$$f_d^{-1}(x) = f_c^{-1}(1 - x)$$

and

$$n_d(x) = f_d^{-1}(1-f_d(x)) = f_d^{-1}(1-(1 - f_c(x))) = f_d^{-1}(f_c(x)) = n(x) = f_c^{-1}(1-f_c(x)) = n_c(x).$$

□

Remark 4.33. Let us suppose that in a connective system the De Morgan property holds. If condition (4.19) holds, then

$$n_c(x) = n(x) = n_d(x),$$

and therefore the system is consistent.

Remark 4.34. Note that if condition (4.19) holds, we get the the classical nilpotent (Łukasiewicz) logic.

4.3.2.1 Bounded systems

The question arises, whether we can use more than one generator functions in our connective system without losing consistency. In the literature only systems generated by only one generator function have been considered, see e.g. Baczyński and Jayaram, [4], Theorem 2.3.18. In these systems the natural negations of the conjunction and the disjunction coincide with the negation operator. Next, the case $n_c(x) \neq n_d(x) \neq n(x)$ is examined.

Definition 4.35. A nilpotent connective system is called a bounded system if

$$f_c(x) + f_d(x) > 1, \text{or equivalently } n_d(x) < n(x) < n_c(x)$$

holds for all $x \in (0, 1)$, where f_c and f_d are the normalized generator functions of the conjunction and disjunction, and n_c, n_d are the natural negations.

The following example shows the existence of consistent bounded systems.

Example 4.8. *(See also Table A.1.) The connective system generated by*

$$f_c(x) := 1 - x^\alpha, f_d(x) := 1 - (1 - x)^\alpha, n(x) := 1 - x, \quad \alpha \in (1, \infty]$$

is a consistent bounded system.

Proof. Applying (4.14) from Proposition 4.23, we obtain: $f_c(n(x)) = 1-(1-x)^\alpha = f_d(x)$, which means that the De Morgan law holds. It is easy to see that $n_c(x) = \sqrt[\alpha]{1-x^\alpha}$, $n_d(x) = 1 - \sqrt[\alpha]{1-(1-x)^\alpha}$, i.e.

$$n_d(x) < n(x) < n_c(x),$$

which means that the classification property is also true (see Figure 4.1).

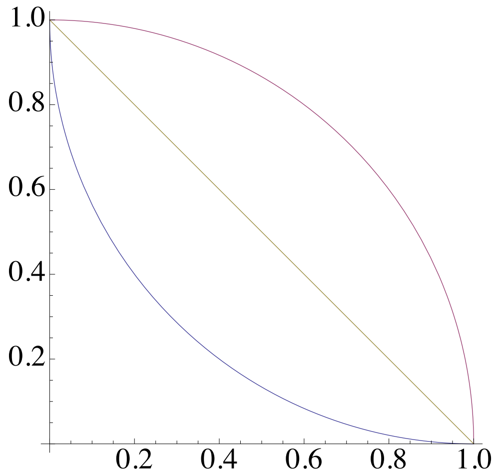


FIGURE 4.1

For the normalized generator functions we have $f_c(x) + f_d(x) > 1$ for all $x \in (0,1)$. $\square$

Remark 4.36. In Example 4.8 for $\alpha = 1$ we get $n_d(x) = n(x) = n_c(x)$, i.e. $f_c(x)+f_d(x) = 1$.

Proposition 4.37. *In a connective system* (c, d, n), *the following statements are equivalent:*

$$f_c(x) + f_d(x) > 1 \quad \textit{for all } x \in (0,1), \tag{4.21}$$

$$f_d\left(f_c^{-1}(x)\right) > 1 - x \quad \textit{for all } x \in (0,1), \tag{4.22}$$

$$f_c\left(f_d^{-1}(x)\right) > 1 - x \quad \textit{for all } x \in (0,1), \tag{4.23}$$

where f_c *and* f_d *are the normalized generator functions of* c *and* d.

Proof. From $n_d(x) < n(x) < n_c(x)$ we have $f_d^{-1}(1 - f_d(x)) < f_c^{-1}(f_d(x))$. Substituting x by $f_d(x)$ we get $f_d^{-1}(1-x) < f_c^{-1}(x)$, i.e. $f_c\left(f_d^{-1}(x)\right) > 1-x$, which is also equivalent to $f_c\left(f_d^{-1}(1-x)\right) > x$. $\square$

Next I consider the case of the rational family of the normalized generator functions introduced by Dombi in [25].

Proposition 4.38. *For the Dombi functions (see also Equation (4.2) and Proposition 4.1)*

$$f_n(x) = \frac{1}{1 + \frac{\nu}{1-\nu}\frac{1-x}{x}}, \quad f_n(0) = 0,$$

$$f_c(x) = \frac{1}{1 + \frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}, \quad f_d(0) = 0,$$

$$f_d(x) = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}, \quad f_c(1) = 0,$$

the following statements are equivalent:

1. *The connective system generated by the Dombi functions in Proposition 4.38 satisfies the De Morgan law.*
2. *For parameters ν_d and ν_c in the normalized generator functions and for parameter ν in the negation function the following equation holds:*

$$\left(\frac{1-\nu}{\nu}\right)^2 = \frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}. \tag{4.24}$$

Proof. By Proposition 4.23, the De Morgan law holds iff:

$$f_c(n(x)) = f_d(x). \tag{4.25}$$

From Proposition 4.1 for $\alpha = -1$ we know that

$$n(x) = \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}}, \tag{4.26}$$

so

$$f_c(n(x)) = \frac{1}{1 + (\frac{\nu_c}{1-\nu_c})(\frac{\nu}{1-\nu})^2 \frac{1-x}{x}} = \frac{1}{1 + \frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}.$$

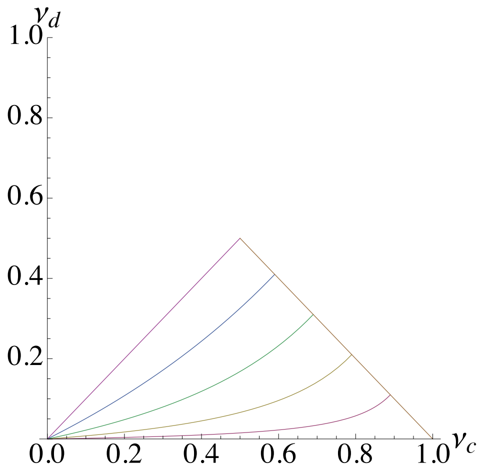
This means that the equality (4.25) holds if and only if the parameters on the left and the right hand side are equal, i.e.:

$$\left(\frac{1-\nu}{\nu}\right)^2 = \frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}. \tag{4.27}$$

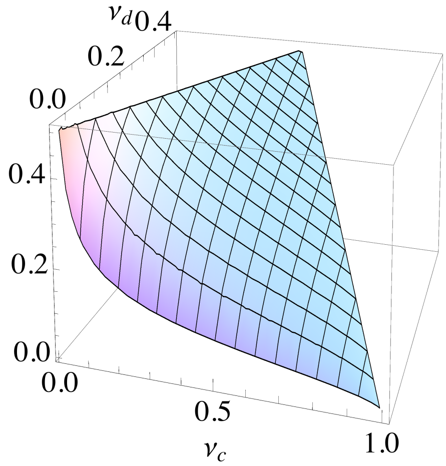
□

Remark 4.39. From (4.27) we get that the De Morgan law holds iff

$$\nu = \frac{1}{1 + \sqrt{\frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}}}. \tag{4.28}$$



(a) The relationship of ν_c and ν_d for different fixed values of ν



(b) ν as a function of ν_c and ν_d

FIGURE 4.2: The relationship between ν, ν_c and ν_d in consistent rational systems

Proposition 4.40. *For the natural negations derived from the Dombi functions defined in Proposition 4.38, the following statements are equivalent for $x \in (0, 1)$:*

$$n_d(x) < n(x) < n_c(x), \tag{4.29}$$

$$\nu_d < \nu < \nu_c. \tag{4.30}$$

Proof.

$$\frac{1}{1 + (\frac{1-\nu_d}{\nu_d})^2 \frac{x}{1-x}} < \frac{1}{1 + \left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}}$$

(see Table A.5) if and only if $\nu_d < \nu$. Similarly, we can prove the other side of the inequality as well. □

Remark 4.41. Note that if the De Morgan property holds,

$$f_c(x) + f_d(x) > 1 \tag{4.31}$$

is also equivalent to (4.29) and (4.30).

Proposition 4.42. *For the Dombi functions defined in Proposition 4.38, the followings are equivalent for $x \in (0, 1)$:*

$$f_c(x) + f_d(x) > 1, \tag{4.32}$$

$$\nu_c + \nu_d < 1. \tag{4.33}$$

Proof.

$$\frac{1}{1+\left(\frac{\nu_c}{1-\nu_c}\frac{x}{1-x}\right)} > 1 - \frac{1}{1+\left(\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}\right)} = \frac{1}{1+\left(\frac{1-\nu_d}{\nu_d}\frac{x}{1-x}\right)}$$

if and only if

$$\frac{\nu_c}{1-\nu_c} < \frac{\nu_d}{1-\nu_d},$$

which is equivalent to $\nu_c + \nu_d < 1$. □

Remark 4.43. Note that if the De Morgan property holds,

$$n_d(x) < n(x) < n_c(x) \tag{4.34}$$

is also equivalent to (4.32) and (4.33).

The relationship between ν_c and ν_d from Propositions 4.40 and 4.42 can be seen in Figure 4.2. In Figure 4.2 we can see the possible values of ν_c and ν_d for fixed values of ν. The values of ν as a function of ν_c and ν_d can be seen on Figure 4.2.

Remark 4.44. By using (4.34), (4.33) and (4.28) we obtain that in a consistent system with

$f_c(x) + f_d(x) > 1$, $\nu < \frac{1}{2}$ always holds.

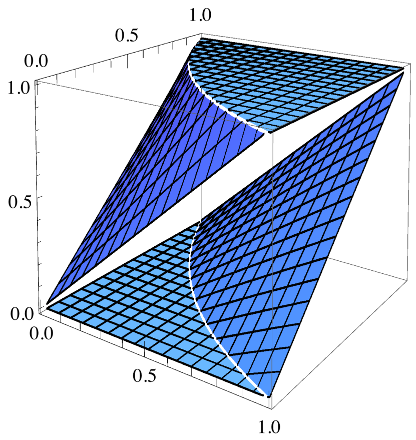
Remark 4.45. For $\nu = \frac{1}{2}$ we get $\sqrt{\frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}} = 1$, so $\nu_c = \nu_d = \nu = \frac{1}{2}$.

Example 4.9. *For* $\nu_c = 0.5$ *and* $\nu_d = 0.1$ $\nu = 0.25$, $\nu_c + \nu_d < 1$ *and* $n_d(x) < n(x) < n_c(x)$.

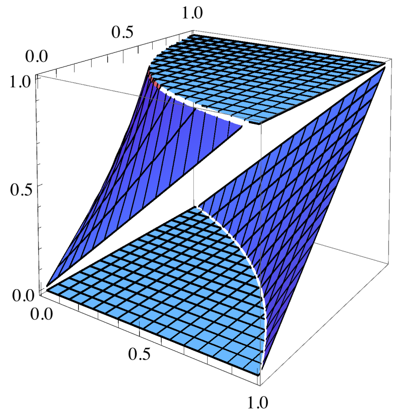
In Figure 4.3 and 4.3 examples for conjunctions and disjunctions are shown for $f_c(x) + f_d(x) = 1$ and for $f_c(x) + f_d(x) > 1$ respectively. Note that the coincidence and the separation of n_c and n_d (see their alternative definition in Proposition 4.13 as well) can easily be seen.

4.4 Overview

Next we give an overview of the three families of normalized generator functions used in our examples and propositions, namely power, exponential and rational functions (see also (4.2), (4.3) and (4.4)). See tables A.1, A.3 and A.4. For the power generator functions the logical connectives are also given, see Table A.2. In the case of the rational and in a special case of the power functions we give the normalized generators in terms of the neutral values as well. Finally, we give some examples of consistent connective systems with mixed types of normalized generator functions, see Table A.6.



(a) $\nu_c = 0.6$ and $\nu_d = 0.4$ ($\nu_c + \nu_d = 1$)



(b) $\nu_c = 0.4$ and $\nu_d = 0.3$ ($\nu_c + \nu_d < 1$)

FIGURE 4.3: Conjunction $c[x, y]$ and disjunction $d[x, y]$

Thesis 2.1.

The concept of a nilpotent connective system is introduced. It is shown that a consistent logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic, which means that nilpotent logical systems are wider than we have thought earlier. Using more than one generator functions, three naturally derived negations are examined. It is shown that the coincidence of the three negations leads back to a system which is isomorphic to Łukasiewicz-logic. Consistent nilpotent logical structures with three different negations are also provided.

Thesis 2.2.

Necessary and sufficient conditions for the classification property (the excluded middle and the law of contradiction), the De Morgan law and consistency have been given.

Chapter 5

Implication Operators in Bounded Systems

5.1 Motivation and scope

In Section 4, it was shown that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-system. Using more than one generator function, consistent nilpotent connective systems could be obtained in a significantly different way with three naturally derived negations. Those consistent nilpotent connective systems which are not isomorphic to Łukasiewicz logic are called bounded systems. Based on the results of Section 4, now I focus on implications in bounded systems.

The results of this chapter can be found in Dombi and Csiszár, [27].

Fuzzy implications are definitely among the most important operations in fuzzy logic [4, 57]. Firstly, other basic logical connectives of the binary logic can be obtained from the classical implication. Secondly, the implication operator plays a crucial role in the inference mechanisms of any logic, like modus ponens, modus tollens, hypothetical syllogism in classical logic. Fuzzy implications all generalize the classical implication with the two possible crisp values from 0, 1, to the fuzzy concept with truth values from the unit interval [0, 1] [82]. In classical logic the implication can be defined in several ways. The most well-known implications are the usual material implication from the Kleene algebra, the implication obtained as the residuum of the conjunction in Heyting algebra (also called pseudo-Boolean algebra) in the intuitionistic logic framework and the implication in the setting of quantum logic. While all these differently defined implications have identical truth tables in the classical case, the natural generalizations of the above definitions in the fuzzy logic framework are not identical. This fact has led to some throughout research on fuzzy implications [2, 5, 6, 7, 8, 48, 64, 69, 70, 75].

Next I focus on residual and S-implication operators in bounded systems. This section is organized as follows. After some preliminaries in Section 5.2, I examine the residual implication in Section 5.3 and S-implications with special attention to the ordering property in Section 5.4. In Section 5.6 I show that in a bounded system, the minimum and maximum operators can also be expressed in terms of the conjunction, the implication and the negation. Finally in Section 5.5 I show that in a bounded system the implications examined in this section can never coincide. The formulae and the properties of implications are summarized in Section 5.7.

5.2 Preliminaries

A mapping $i : [0,1]^2 \to [0,1]$ is called an implication operator if and only if it satisfies the boundary conditions $i(0,0) = i(0,1) = i(1,1) = 1$ and $i(1,0) = 0$.

The above conditions are the minimum requirements for an implication operator. Other potentially interesting properties of implication operators are listed in [4, 14, 33, 69, 75]. All fuzzy implications can be obtained by generalizing the implication operator of classical logic. In this sense, Fodor and Roubens, [38], established the following definition.

Definition 5.1. A fuzzy implication is a function $i : [0,1]^2 \to [0,1]$ that satisfies the following properties:

1. The first place antitonicity:

$$\text{for all } x_1, x_2, y \in [0,1] \quad (if\ x_1 \leq x_2\ then\ i(x_1, y) \geq i(x_2, y)). \tag{FA}$$

2. The second place isotonicity:

$$\text{for all } x, y_1, y_2 \in [0,1] \quad (if\ y_1 \leq y_2 \text{ then } i(x, y_1) \leq i(x, y_2)). \tag{SI}$$

3. The dominance of falsity of antecedent:

$$i(0, y) = 1 \quad \text{for all} \quad y \in [0,1]. \tag{DF}$$

4. The dominance of truth of consequent:

$$i(x, 1) = 1 \quad \text{for all} \quad x \in [0,1]. \tag{DT}$$

5. The boundary condition:

$$i(1,0) = 0 \quad \text{and} \quad i(1,1) = 1. \tag{BC}$$

Other important but usually not required properties of fuzzy implications are defined below (see Baczyński and Jayaram, [4]).

Definition 5.2. A fuzzy implication i satisfies

1. The left neutrality property (the neutrality of truth) if

$$i(1,y) = y \quad \text{for all} \;\; y \in [0,1]. \tag{LN}$$

2. The exchange principle if

$$i(x, i(y,z)) = i(y, i(x,z)) \quad \text{for all} \;\; x, y, z \in [0,1]. \tag{EP}$$

3. The identity principle if

$$i(x,x) = 1 \quad \text{for all} \;\; x \in [0,1]. \tag{IP}$$

4. The strong negation principle if the mapping n^* defined as

$$n^*(x) = i(x,0) \quad \text{for all} \;\; x \in [0,1] \tag{SN}$$

   is a strong negation.

5. The law of contraposition (or in other words,the contrapositive symmetry) with respect to a strong negation n if

$$i(x,y) = i(n^*(y), n^*(x)) \quad \text{for all} \;\; x, y \in [0,1]. \tag{LC}$$

6. The ordering property if

$$i(x,y) = 1 \quad \text{if and only if} \quad x \le y \quad \text{for all} \;\; x, y \in [0,1]. \tag{OP}$$

Remark 5.3. The negation operator n^* is also called the natural negation of the implication i (see Baczyński and Jayaram, [4]).

A detailed study of possible relations between all these properties can be found in [4, 14, 70]. Notice that other properties can also be found in the literature. In particular, $i(x, n^*(x)) = n^*(x)$ for all $x \in [0,1]$, where n^* is a strong negation (see Mas et al., [57]).

Three well-established classes of implication operators are (S,N)-, QL- and R-implications.

Definition 5.4. (Baczyński and Jayaram, [4], page 57.) A function $i : [0,1]^2 \rightarrow [0,1]$ is called an S-implication if there exists a t-conorm S and a strong negation n^* such that

$$i_S(x,y) = S(n^*(x),y), \quad x,y \in [0,1].$$

Definition 5.5. (Baczyński and Jayaram, [4], page 90.) A function $i : [0,1]^2 \rightarrow [0,1]$ is called a QL-operation if there exists a t-conorm S, a t-norm T and a strong negation n^* such that

$$i_Q(x,y) = S(n^*(x), T(x,y)), x,y \in [0,1].$$

In general, QL-operations violate property (FA). For the conditions under which (FA) is satisfied see Fodor, [37]. When a QL-operation is a fuzzy implication, then it is called a QL-implication.

Definition 5.6. (Baczyński and Jayaram, [4], page 68.) A function $i : [0,1]^2 \rightarrow [0,1]$ is called an R-implication if there exists a t-norm T such that

$$i_R(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\}.$$

In the case where the given t-norm is left-continuous, I will refer to the R-implication defined above as a residual implication [4, 48, 50]. Note that in this case we have $T(x,y) = \underset{z}{inf}\,(z \in [0,1], \mid i(x,z) \geq y)$. It is easy to see that both S-implications and R-implications satisfy properties (FA)-(BC), regardless of the t-norm T, the t-conorm S and the strong negation n^* types. Hence, they are implications in the Fodor and Roubens sense. Different characterizations of S-implications, QL-implications and R-implications can be found in the literature (for details, see [5, 35, 38]). It is worth mentioning here that new characterizations of R and S-implications can also be found at Trillas, [76].

Next, implications in bounded systems are to be examined.

5.3 R-implications in bounded systems

For implications in nilpotent connective systems notation i is used. For the residual implication, we easily get the following formula (see Theorem 2.5.21., Baczyński and Jayaram, [4]).

Proposition 5.7. *In a nilpotent connective system* (c, d, n) *the residual implication has the following form.*

$$i_R(x, y) = f_c^{-1}\big[f_c(y) - f_c(x)\big],$$

where f_c *is the additive generator function of* c*, and* $[\ ]$ *is the cutting operator defined in Definition 1.9.*

Proof. From the definition of residual implication,

$$i_R(x, y) = max\,\{z : \ c(x, z) \le y\}\,,$$

where

$$c(x, z) = f_c^{-1}\big[f_c(x) + f_c(z)\big] \le y.$$

From this, we have $z = f_c^{-1}\big[f_c(y) - f_c(x)\big]$. □

Proposition 5.8. *We can also express* i_R *by using the negation operator and the normalized additive generator function of* d*.*

Proof. From $n(x) = f_c^{-1}(f_d(x))$, we have

$$f_c(x) = f_d(n(x)) \quad \text{and} \quad f_c^{-1}(x) = n^{-1}\left(f_d^{-1}(x)\right),$$

$$i_R(x, y) = n^{-1}\big(f_d^{-1}\left[f_d(n(x)) - f_d(n(y))\right]\big).$$

□

The notation H is introduced below for further applications. A new formula for i_R is given in (5.2) by using H.

$$H(x) = 1 - f_d(n(x)), \tag{5.1}$$

so $\quad n^{-1}\left(f_d^{-1}(x)\right) = H^{-1}(1 - x)$. From this we have

$$i_R(x, y) = H^{-1}\left(1 - \big[1 - H(x) - (1 - H(y))\big]\right) = H^{-1}\left[H(y) - H(x) + 1\right]. \tag{5.2}$$

Next, the properties in Definition 5.2 are examined to see whether they are compatible with the R-implication in a nilpotent connective system.

Remark 5.9. Note that the following results regarding the properties of i_R correspond with Section 2.5. at Baczyński and Jayaram, [4].

Proposition 5.10. *In a nilpotent connective system, i_R satisfies*

1. *the **left neutrality property** (the neutrality of truth), (**LN**) i.e.* $i_R(1, y) = y$ *for all* $y \in [0, 1]$,
2. *the **exchange principle**, (**EP**) i.e.* $i_R(x, i_R(y, z)) = i_R(y, i_R(x, z))$ *for all* $x, y, z \in [0, 1]$,
3. *the **identity principle**, (**IP**) i.e.* $i_R(x, x) = 1$ *for all* $x \in [0, 1]$,
4. *the **strong negation principle**, (**SN**), since* $n_R^*(x) = i_R(x, 0) = n_c(x)$ *for all* $x, y \in [0, 1]$ *is a strong negation,*
5. *the **law of contraposition** (contrapositive symmetry), (**LC**) with respect to the strong negation in (**SN**); i.e.* $i_R(x, y) = i_R(n_c(y), n_c(x))$ *for all* $x, y \in [0, 1]$,
6. *the **ordering principle**, (**OP**) is valid for* $i_R(x, y)$*, i. e.* $i_R(x, y) = 1$ *if and only if* $x \leq y$.

Proof. LN, EP, IP and OP always hold for an R-implication derived from a continuous t-norm (see Theorem 2.5.7, Baczyński and Jayaram, [4]).

LC follows directly from the definition of n_c.

EP and OP together always imply SN for continuous implications (see Corollary 1.4.19, Baczyńskiand Jayaram, [4]).

□

Remark 5.11. Note that the **law of contraposition** (contrapositive symmetry), (**LC**) with respect to the strong negation n; i.e. $i_R(x, y) = i_R(n(y), n(x))$ for all $x, y \in [0, 1]$, never holds in a bounded system (see also Corollary 1.5.12., Baczyński and Jayaram, [4]).

Proof. I prove that $i_R(x, y) = i_R(n(y), n(x))$ holds for all $x, y \in [0, 1]$ if and only if $f_c(x) + f_d(x) = 1$; i.e. the system is a Łukasiewicz logical system.

If $x \leq y$, then $n(y) \leq n(x)$, and therefore from the ordering property we get that both sides are equal to 1.

If $x > y$, then the two sides of the equality are equal if and only if
$f_c(y) - f_c(x) = f_d(x) - f_d(y)$, i.e. $f_c(x) + f_d(x) = f_c(y) + f_d(y)$
for all $x, y \in [0, 1]$, which means that $f_c(x) + f_d(x)$ is constant.

Since $f_c(0) + f_d(0) = 1, \quad f_c(x) + f_d(x) = 1.$ □

A different form of the residual implication is also given in the following section.

5.4 S-implications in bounded systems

In a nilpotent connective system (c, d, n) we can define different types of S-implications.

Definition 5.12.

1. $i_{S_n}(x, y) = d(n(x), y), \quad x, y \in [0, 1],$
2. $i_{S_d}(x, y) = d(n_d(x), y), \quad x, y \in [0, 1],$
3. $i_{S_c}(x, y) = d(n_c(x), y), \quad x, y \in [0, 1],$

where n_c and n_d are the natural negations of c and d.

Replacing the disjunction in the definitions above by an appropriate composition of negations and the conjunction leads us to further possible definitions of implications. Since in a bounded system the negations n, n_c and n_d never coincide, negations different from n can also be used similarly to the De Morgan identity.

Definition 5.13. In a nilpotent connective system (c, d, n)

1. $i^c_{S_n}(x, y) = n\left(c(x, n(y))\right), \quad x, y \in [0, 1],$
2. $i^c_{S_d}(x, y) = n_d\left(c(x, n_d(y))\right), \quad x, y \in [0, 1],$
3. $i^c_{S_c}(x, y) = n_c\left(c(x, n_c(y))\right), \quad x, y \in [0, 1].$

where n_c and n_d are the natural negations of c and d.

Note that from the De Morgan identity it follows immediately that
$i^c_{S_n}(x, y) = i_{S_n}(x, y)$ and as the following proposition shows, $i^c_{S_c}$ is the residual implication.

Proposition 5.14. *In a nilpotent connective system* (c, d, n)
$i^c_{S_c}(x, y) = f_c^{-1}[f_c(y) - f_c(x)] = i_R(x, y)$, *where* f_c *is the normalized additive generator function of* c.

Proof.

$$\begin{aligned} i^c_{S_c}(x,y) &= n_c\left(c(x,n_c(y))\right) = \\ &= n_c\left(f_c^{-1}\left[f_c(x)+1-f_c(y)\right]\right) = \\ &= f_c^{-1}[1-(1-f_c(y)+f_c(x))] = \\ &= f_c^{-1}[f_c(y)-f_c(x)]. \end{aligned}$$

□

5.4.1 Properties of i_{S_n}, i_{S_d} and i_{S_c}

First the formulae for the S-implications defined above are given.

Proposition 5.15. *In a nilpotent connective system* (c,d,n)

1. $i_{S_n}(x,y) = f_d^{-1}[f_c(x)+f_d(y)]$,
2. $i_{S_d}(x,y) = f_d^{-1}[1-f_d(x)+f_d(y)]$,
3. $i_{S_c}(x,y) = f_d^{-1}\left[f_d(y)+f_d(n_c(x))\right]$,

where f_c *and* f_d *are the normalized additive generator functions of* c *and* d*, respectively.*

Proof. All the three formulae are easy to verify. □

Next, the basic properties of the S-implications in a nilpotent connective system are stated. Note that the following results are consistent with those described in Section 2.5, Baczyński and Jayaram, [4].

Proposition 5.16. *In a nilpotent connective system,* i_{S_n}, i_{S_d} *and* i_{S_c} *satisfy*

1. *the* ***left neutrality property*** *(the neutrality of truth),* (***LN***)*, i.e.* $i(1,y)=y$ *for all* $y\in[0,1]$,
2. *the* ***exchange principle****,* (***EP***)*, i.e.* $i\left(x,i(y,z)\right) = i\left(y,i(x,z)\right)$ *for all* $x,y,z\in[0,1]$,
3. *the* ***identity principle****,* (***IP***)*, i.e.* $i\left(x,x\right)=1$ *for all* $x\in[0,1]$,
4. *the* ***strong negation*** *principle ,* (***SN***) *since* $i_S\left(x,0\right)$ *for all* $x,y\in[0,1]$ *is a strong negation,*

5. *the* ***law of contraposition*** *(contrapositive symmetry), (**LC**) with respect to the strong negation in SN.*

Proof.

1. LN holds for every S-implication (see Proposition 2.4.3, Baczyński and Jayaram, [4]),

2. EP holds for every S-implication (see Proposition 2.4.3, Baczyński and Jayaram, [4]),

3. IP holds as well, because of the consistency property and the use of nilpotent operators (see Theorem 2.4.17, Baczyński and Jayaram, [4]).

4. For SN,

   (a) $n_n^*(x) = i_{S_n}(x,0) = d(n(x),0) = f_d^{-1}\left[f_d(n(x)) + 0\right] = n(x)$,

   (b) $n_d^*(x) = i_{S_d}(x,0) = d(n_d(x),0) = f_d^{-1}\left[f_d(n_d(x)) + 0\right] = n_d(x)$,

   (c) $n_c^*(x) = i_{S_c}(x,0) = d(n_c(x),0) = f_d^{-1}\left[f_d(n_c(x)) + 0\right] = n_c(x)$,

5. LC is trivial.

□

5.4.2 S-implications and the ordering property

First, the so-called weak ordering principle for implications is defined. Although the ordering principle plays an important role, as we will see, only the weak ordering property can be required in general.

Definition 5.17. The implication i satisfies the **weak ordering principle (WOP)** if the following statement holds:

$$i(x,y) = 1 \quad \text{if and only if} \quad x \le \tau(y),$$

where τ is a strictly increasing function from $[0,1] \to [0,1]$ with $\tau(0) = 0$ and $\tau(1) = 1$.

Remark 5.18. In the terminology of Maes and De Baets, τ from Definition 5.17 is an affirmation (see Maes and De Baets, [55]).

Remark 5.19. Note that for $\tau(x) = x$ we get the original ordering property (OP).

Henceforth I use the following notations for the composition of two negation operators.

Definition 5.20. In a connective system (c, d, n)

$$\tau_{n,d}(x) := n(n_d(x)),$$

and

$$\tau_{c,d}(x) := n_c(n_d(x)),$$

where n_c and n_d are the natural negations of c and d respectively.

Remark 5.21. Note that in a consistent connective system $\tau_{d,n} = \tau_{n,c}$ and similarly, $\tau_{c,n} = \tau_{n,d}$.

Proposition 5.22. *In a nilpotent connective system i_{S_d} satisfies the **ordering principle (OP)**, while i_{S_n} and i_{S_c} satisfy the **weak ordering principle (WOP)**.*

Proof.

For i_{S_d} we have the following:
$i_{S_d}(x, y) = 1$ if and only if $f_d^{-1}\left[f_d(n_d(x)) + f_d(y)\right] = 1$,
which means that $f_d(n_d(x)) + f_d(y) \geq 1$, from which we get $n_d(x) \geq n_d(y)$, which holds if and only if $x \leq y$.

For i_{S_c}, let $\tau(x) = \tau_{c,d}(x) = n_c(n_d(x))$.
$i_{S_c}(x, y) = 1$ if and only if $f_d^{-1}\left[f_d(n_c(x)) + f_d(y)\right] = 1$,
which means that $f_d(n_c(x)) + f_d(y) \geq 1$, from which we get $n_c(x) \geq n_d(y)$, so $x \leq n_c(n_d(y)) = \tau_{c,d}(y)$.

Similarly, for i_{S_n}, let $\tau(x) = \tau_{n,d}(x) = n(n_d(x))$.
$i_{S_n}(x, y) = 1$ if and only if $f_d^{-1}\left[f_d(n(x)) + f_d(y)\right] = 1$,
which means that $f_d(n(x)) + f_d(y) \geq 1$, from which we get $n(x) \geq n_d(y)$, so $x \leq n(n_d(y)) = \tau_{n,d}(y)$. □

Next I give an example for a bounded system illustrating that i_{S_n} does not satisfy the ordering property. For $f_c(x) = 1 - x^2; f_d(x) = 1 - (1 - x)^2$;
$n(x) = 1 - x$, there exist an x and a y for which $i_{S_n}(x, y) = 1$ and $y < x$, i.e. the ordering principle does not hold, because $i_{S_n}(x, y) = 1$ if and only if
$d(n(x), y) = 1$.

For $x = 0.5$ and $y = 0.4$ we get
$f_c(0.5) + f_d(0.4) = (1 - 0.5^2) + (1 - (1 - 0.4)^2) = 0.75 + (1 - 0.36) = 1.39$,
so $i(0.5, 0.4) = 1$ and $(y < x)$.

Remark 5.23. Note that the following statements are equivalent:

$$i_{S_c}(x, y) = 1 \quad \text{if and only if} \quad x \leq y \tag{5.3}$$

$$f_c(x) + f_d(x) = 1 \quad \text{for all } x \in [0, 1]. \tag{5.4}$$

In other words, the **ordering property**, (**OP**) never holds in a bounded system.

I show that the ordering property holds if and only if $f_c(x) + f_d(x) = 1$. We have $n_c(x) \geq n_d(y)$. This means that the ordering property for i_{S_c} (and also similarly for i_{S_n}) is equivalent to the followings: $n_c(x) \geq n_d(y) \quad$ if and only if
$x \leq y$.

It is easy to see that the condition above holds if and only if $n_d(x) = n_c(x)$, i.e. $f_c(x) + f_d(x) = 1$.

5.5 Comparison of implications in bounded systems

Now I prove that in a bounded system, the different types of implications considered so far never coincide.

Proposition 5.24. *In a connective system (c, d, n), any two of the implications defined so far coincide if and only if $f_c(x) + f_d(x) = 1$, where f_c and f_d are the normalized additive generator functions of c and d respectively.*

Proof. It was shown in Section 4 that in a bounded system (where n_c, n_d and n are different) the natural negations of the implications in question are the same only in the case of i_R and i_{S_d}, which simply means that it is sufficient to examine their equality.

Since i_R satisfies OP while i_{S_c} for $f_c(x) + f_d(x) \neq 1$ does not (see Table A.7), we see that in a bounded system they cannot be equal.

□

Remark 5.25. It is clear that in a Łukasiewicz logical system
(where $f_c(x) + f_d(x) = 1$), all the implications considered in this section coincide.

From the results of sections 5.3 and 5.4, we can say that in a bounded system we have two different implications (namely i_R and i_{S_d}) that satisfy all of the properties $LN-OP$ (see Table A.7). Hence, the notations i_c and i_d are used, to coincide with the additive generator functions f_c and f_d used in the formulae of the implications, respectively (see Table A.7). Henceforth let us use the following notation for sake of simplicity.

$$i_d(x,y) := i_{S_d}(x,y) \text{ and } i_c(x,y) := i_R(x,y). \tag{5.5}$$

5.6 Min and Max operators in nilpotent connective systems

In this section I show that in a nilpotent connective system, the minimum and maximum operators can be expressed in terms of the conjunction, the disjunction and the negation.

Proposition 5.26. $c\,(x, i_c(x,y)) = Min(x,y), \quad x,y \in [0,1]$

Proof. $c\,(x, i_c(x,y)) = f_c^{-1}\left[f_c(x) + [f_c(y) - f_c(x)]\right].$

For $x \le y$ $f_c(x) \ge f_c(y)$, which means that $c\,[x, i_c(x,y)] = x$.

Similarly, for $x \ge y$ $f_c(x) \le f_c(y)$, which means that $c\,(x, i_c(x,y)) = y$.

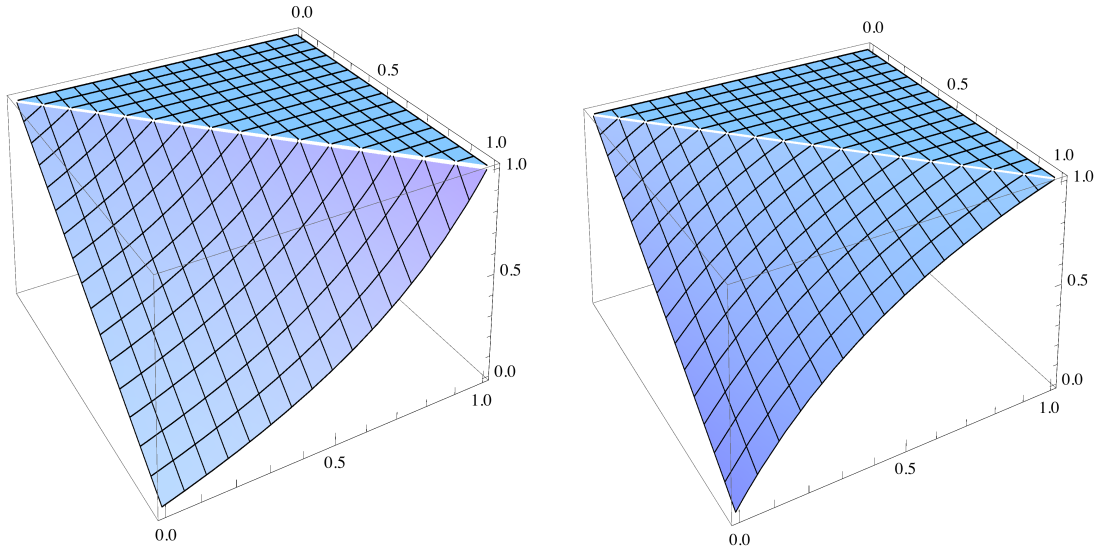
□

Proposition 5.27. $n\,(c\,(n(x), i_c\,(n(x), n(y)))) = Max\,(x,y), \quad x,y \in [0,1]$

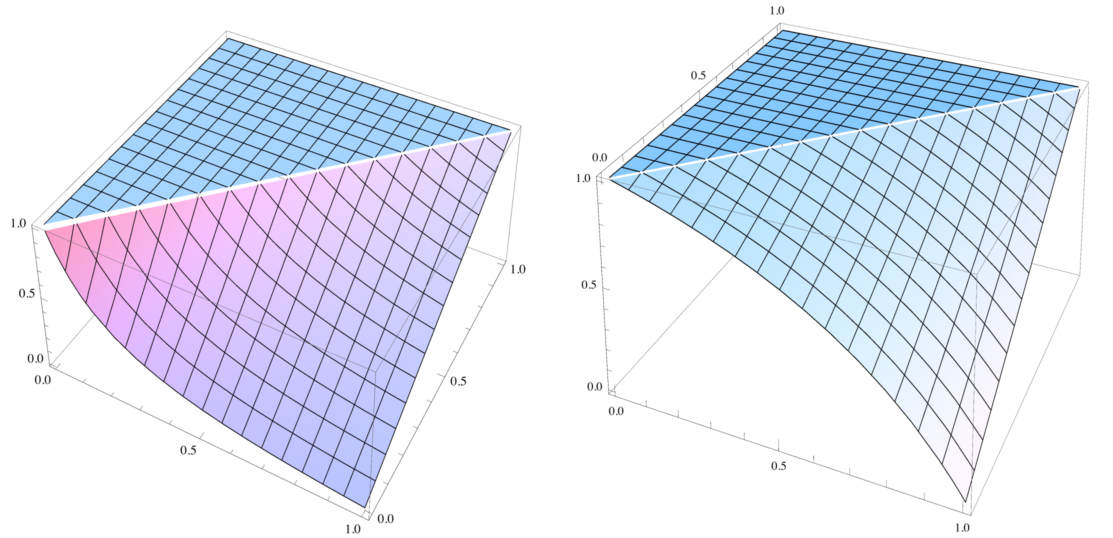
Proof. The statement follows immediately from the previous proposition (or also can been proved similarly). □

5.7 Overview

In Table A.7, the results concerning the properties of each implication are listed. For rational additive generator functions, the implications have been plotted in Figures 5.1 to 5.3. The formulae of the additive generators and the implications are summarized in Tables A.7 and A.4.

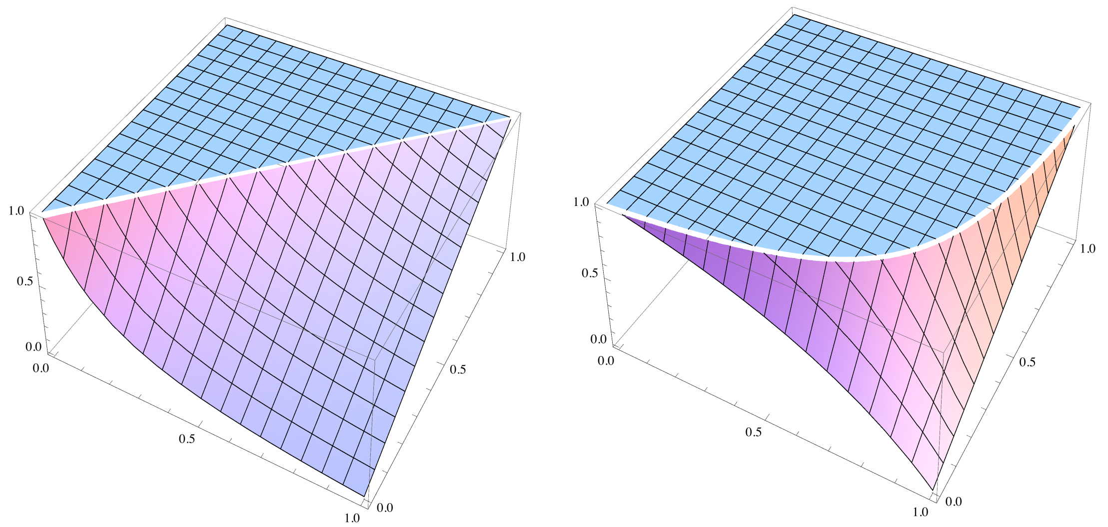


(a) i_c for $\nu_c = 0.4$ (b) i_c for $\nu_c = 0.6$



(c) i_d for $\nu_d = 0.3$ (d) i_d for $\nu_d = 0.6$

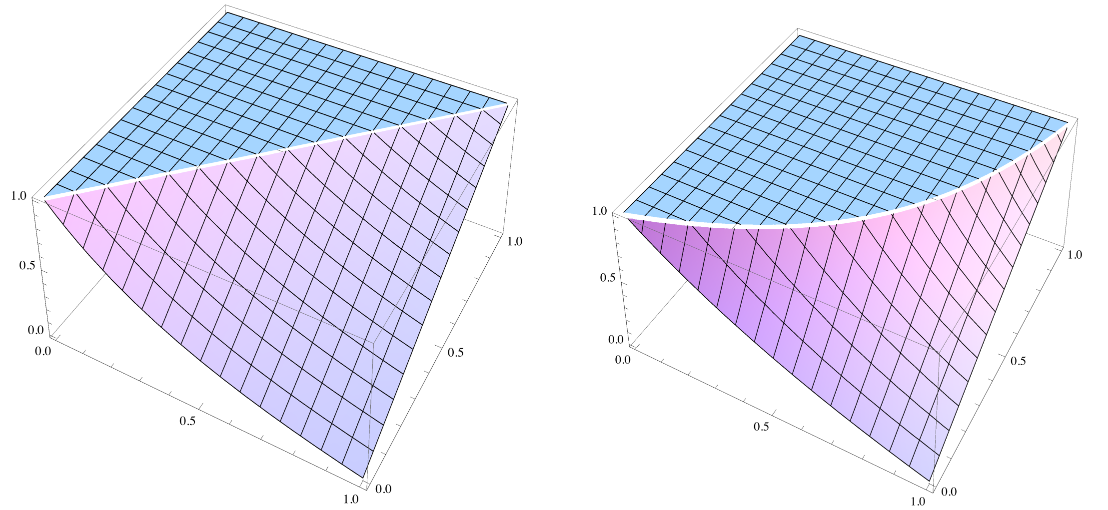
FIGURE 5.1: i_c and i_d implications for rational generators



(a) $\nu_c = 0.7$ and $\nu_d = 0.3$

(b) $\nu_c = 0.4$ and $\nu_d = 0.3$

FIGURE 5.3: S_c-implications for rational generators



(a) $\nu_c = 0.6$ and $\nu_d = 0.4$

(b) $\nu_c = 0.4$ and $\nu_d = 0.3$

FIGURE 5.2: S_n-implications for rational generators

Thesis 2.3.

Both R- and S-implications with respect to the three naturally derived negations of the bounded system are considered. It is shown that these implications never coincide in a bounded system, as the condition of coincidence is equivalent to the coincidence of the negations, which would lead to Łukasiewicz logic. The formulae and the basic properties of four different types of implications are given, two of which fulfill all the basic properties generally required for implications. A wide range of examples is also presented. The concept of a weak ordering property is defined. Two different implications, i_c and i_d are introduced, both of which fulfill all the basic features generally required for implications.

Chapter 6

Equivalence Operators in Bounded Systems

6.1 Motivation and scope

The theory of fuzzy relations is a generalization of that of crisp relations of a set. Zadeh introduced the concept of fuzzy relations in [82] and the concept of fuzzy similarity relations in [83]. Since then, many authors studied fuzzy equivalence relations [15, 16, 60, 61] and it has proven to be useful in different contexts such as fuzzy control, approximate reasoning and fuzzy cluster analysis. As the research progressed, it became clear that any given relation may or may not satisfy a particular requirement for the fuzzy equivalence relation introduced by Zadeh.

As shown by Gupta and Gupta, [44], the condition $\mu(x, x) = 1$ for $\forall x \in X$ is too strong for defining a fuzzy reflexive relation μ on a set X (see also Yeh, [81] and Chon, [17]). Therefore, new types of fuzzy reflexive relations were needed to be introduced. Yeh, [81], defined the concept of ϵ-reflexive fuzzy relations and weakly reflexive fuzzy relations by weakening the standard reflexive fuzzy relation to $\mu(x, x) \geq \epsilon > 0$. Gupta and Gupta, [44], introduced G-reflexive fuzzy relations as a generalization of reflexive fuzzy relations.

While discussing fuzzy transitive relations, different approaches have been adopted. The first type of transitivity is that introduced by Zadeh in [83], and the second type of transitivity is the so-called T-transitivity of fuzzy relations, defined with the help of the t-norm. In [12, 13, 18, 24], fuzzy T- transitivity has been deeply studied. Recently, Mesiar et al., [59], have noticed that the associativity of a t-norm is superfluous in the above context, especially since we never have to aggregate more than two arguments. Thus, they have substituted a conjunctor instead of a t-norm. An alternative approach based on implications has been considered by Schmechel and Thiele, [67, 72]. Jayaram and Mesiar, [47], studied I-transitivity, where the implicator I is nothing more than a binary operator satisfying the boundary conditions of an implication. Another type of transitivity, the so-called ϵ-fuzzy transitivity, has been introduced by Beg and Ashraf, [9]. Ali et al., [3], introduced the concept of (α, β)-fuzzy reflexive relations, as a generalization of fuzzy reflexive relation as well as of fuzzy G-reflexive relations. More general types of fuzzy symmetric relation, a (α, β)-fuzzy symmetric relation and (α, β)-fuzzy transitive relations, were also studied. The concepts of (α, β)-fuzzy reflexive, symmetric and transitive relations naturally lead to the concept of (α, β)-fuzzy equivalence relations on a set. De Baets and Mesiar, [23], introduced the concept of a T-partition as a generalization of that of a classical partition.

Although the mentioned list of authors is by no means complete, it gives us a slight idea about the importance of the concept of fuzzy equivalence relations in different contexts. Now I resolve a paradox of the equivalence relation by aggregating the implication-based equivalence and its dual operator.

In Section 4, it was shown that a consistent connective system generated by nilpotent operators is not necessarily isomorphic to the Łukasiewicz system. Using more than one generator function, consistent nilpotent connective systems can be obtained in a significantly different way with three naturally derived negation operators. As the class of nilpotent t-norms has preferable properties that make them useful in constructing logical structures, the advantages of such systems are obvious (see Klement and Navara, [50]). Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm (see Grabisch et al., [43]), the nilpotent systems studied earlier were all isomorphic to the well-known Łukasiewicz logic. Those consistent connective systems which are not isomorphic to Łukasiewicz logic are called bounded systems (see Dombi and Csiszár, [27]). Based on the results of Section 4 and 5, my focus is now on equivalences in bounded systems.

This section is organized as follows. After some preliminaries in Section 6.2, I define and examine the implication-based equivalences in bounded systems in Section 6.3. Next, the so-called dual equivalences are introduced and examined in Section 6.4. Using the arithmetic mean operator examined in Section 6.5, the aggregated equivalences are introduced and studied in Section 6.6. I show that unlike the other two types, the aggregated equivalences are threshold transitive and associative as well. In Section 6.7, for further applications as in image processing, the overall equivalence of two grey level images was defined, and an important semantic meaning of the aggregated equivalences was given. Finally, in Section 6.8, I summerize my key results.

The results of this chapter can be found in [28].

6.2 Preliminaries

There exist several approaches to the definition of equivalences. Equivalences can be considered as binary relations [12, 15, 16, 17, 60, 61]. Given a non-empty set X, a subset σ of $X \times X$ is called a binary relation on X. A binary relation σ on X is reflexive if $(x, x) \in \sigma, \forall x \in X$; σ is symmetric if $(x, y) \in \sigma$ implies $(y, x) \in \sigma$, $\forall x, y \in X$; σ is transitive if $(x, y) \in \sigma$ and $(y, z) \in \sigma$ imply $(x, z) \in \sigma$, $\forall x, y, z \in X$. A binary relation is called an equivalence relation if it is reflexive, symmetric and transitive. Recall that a fuzzy subset μ of X is a mapping $\mu : X \to [0, 1]$.

Definition 6.1. (Murali, [60]) A fuzzy binary relation on X and Y is a fuzzy subset μ of $X \times Y$. A fuzzy binary relation on a set X is a fuzzy subset μ of $X \times X$, i.e. a function $\mu : X \times X \to [0, 1]$.

Definition 6.2. (Murali, [60]) A fuzzy relation μ on a set X is said to be reflexive if $\mu(x, x) = 1$, $\forall x \in X$, and symmetric if $\mu(x, y) = \mu(y, x)$, $\forall x, y \in X$.

Definition 6.3. (Zadeh, [83]) A fuzzy relation μ on a set X is said to be fuzzy transitive if

$$\mu(x, z) \geq \sup_{y \in X} \{\min(\mu(x, y), \mu(y, z))\} \quad \forall (x, y), (y, z) \in X \times X.$$

Definition 6.4. (Ali et al.)[3] A fuzzy relation μ on X is a fuzzy equivalence relation if it is a reflexive, symmetric and fuzzy transitive relation on X.

Now I consider an equivalence as a connective. I give the definition of an equivalence as a binary operation on the unit interval according to Fodor and Roubens.

Definition 6.5. (Fodor and Roubens, [38]) A function $e : [0, 1]^2 \to [0, 1]$ is called equivalence if it satisfies the following conditions:

1. Symmetry, i.e. $e(x, y) = e(y, x)$ for $\forall x, y \in [0, 1]$,
2. Compatibility, i.e. $e(0, 1) = e(1, 0) = 0$ and $e(0, 0) = e(1, 1) = 1$,
3. Reflexivity, i.e. $e(x, x) = 1$ for $\forall x \in [0, 1]$,
4. Monotonicity, i.e. $x \leq x' \leq y' \leq y \Rightarrow e(x, y) \leq e(x', y')$.

Definition 6.6. An operator $e(x, y) : [0, 1]^2 \to [0, 1]$ is said to be

1. T-transitive with respect to a t-norm T, if $\forall x, y, z \in [0, 1] : T(e(x, y), e(y, z)) \leq e(x, z)$,

2. threshold transitive with respect to a threshold ν $(0 < \nu < 1)$, if $e(x,y) \geq \nu$ and $e(y,z) \geq \nu$ together imply $e(x,z) \geq \nu$ for $\forall x,y,z \in [0,1]$,
3. invariant with respect to a negation n, if $e(x,y) = e(n(x),n(y))$ for $\forall x,y \in [0,1]$,
4. associative, if $e(x,e(y,z)) = e(e(x,y),z))$ holds for $\forall x,y,z \in [0,1]$.

6.3 Equivalences in bounded systems

Let us now consider a nilpotent connective system (c,d,n) (see Section 4.3) and let us denote the normalized generator functions of c and d by f_c and f_d, respectively. Using the above-defined implications i_c and i_d, we can define two different types of equivalences.

Definition 6.7. The conjunctive and disjunctive equivalence operators are defined as follows.

$$e_c(x,y) = c\left(i_c(x,y), i_c(y,x)\right)$$

$$e_d(x,y) = n_d\left(d\left(n_d\left(i_d(x,y)\right), n_d\left(i_d(y,x)\right)\right)\right)$$

Proposition 6.8. *In a bounded system,*

$$e_c(x,y) = f_c^{-1}\left[|f_c(x) - f_c(y)|\right]$$

and similarly,

$$e_d(x,y) = f_d^{-1}\left[1 - |f_d(x) - f_d(y)|\right].$$

Proof.

$$e_c(x,y) = f_c^{-1}\left[[f_c(y) - f_c(x)] + [f_c(x) - f_c(y)]\right].$$

If $x < y$, then $f_c(x) \geq f_c(y)$, which means that we have $f_c^{-1}\left[f_c(x) - f_c(y)\right]$. Similarly, if $y > x$, then $f_c(x) \leq f_c(y)$ and we get $f_c^{-1}\left[f_c(y) - f_c(x)\right]$. Similarly for e_d, by using $n_d\left(i_d(y,x)\right) = f_d^{-1}\left[f_d(y) - f_d(x)\right]$, we obtain

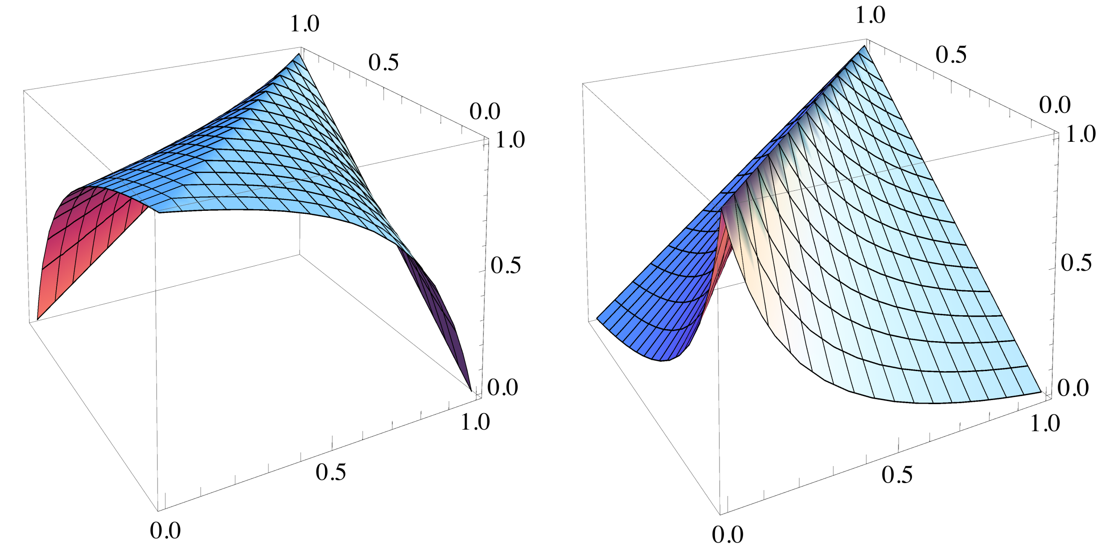
$$n_d(e_d(x,y)) = f_d^{-1}\left[[f_d(x) - f_d(y)] + [f_d(y) - f_d(x)]\right] = f_d^{-1}\left[|f_d(x) - f_d(y)|\right].$$

Therefore,

$$e_d(x,y) = f_d^{-1}\left[1 - |f_d(x) - f_d(y)|\right].$$

□

Remark 6.9. Since $0 \leq |f_c(x) - f_c(y)| \leq 1$ and $0 \leq 1 - |f_d(x) - f_d(y)| \leq 1$, the cutting function can be omitted here. For conceptual reasons, I prefer to leave it in all of the formulae.



(a) e_c with $\nu_c = 0.3$ (b) e_d with $\nu_d = 0.3$

FIGURE 6.1: $e_c(x, y)$ and $e_d(x, y)$ for rational generators

6.3.1 Properties of $e_c(x, y)$ and $e_d(x, y)$

Next, I will examine the chief properties of $e_c(x, y)$ and $e_d(x, y)$ and show that they coincide if and only if the connective system is a Łukasiewicz system.

Proposition 6.10. *Let ν_c and ν_d be the fixpoints of n_c and n_d respectively. The operators, $e_c(x, y)$ and $e_d(x, y)$ have the following properties:*

1. *Compatibility (see Definition 6.5).*
2. *Symmetry (see Definition 6.5).*
3. *Reflexivity (see Definition 6.5).*
4. *Monotonicity (see Definition 6.5).*
5. *e_c is T-transitive with respect to the conjunction c (see Definition 6.6) and similarly, e_d is T-transitive with respect to the t-norm generated by $1 - f_d(x)$.*
6. *e_c and e_d are not threshold transitive (see Definition 6.6) with respect to ν_c and ν_d.*
7. *Invariance (see Definition 6.6) with respect to n_c and n_d.*
8. *$e_c(1, x) = e_c(1, x) = x$, $e_d(0, x) = n_d(x)$, and similarly, $e_c(0, x) = n_c(x)$.*
9. *$e_c(x, y) = 0$ if and only if $x, y \in 0, 1$ and $x \neq y$. Similarly, $e_d(x, y) = 0$ if and only if $x, y \in 0, 1$ and $x \neq y$.*

10. $n_d(e_d(x,y)) = e_d(n_d(x),y))$ *if and only if* $x \in \{0,1\}$ *or* $y \in \{0,1\}$ *and* $n_c(e_c(x,y)) = e_c(n_c(x),y))$ *if and only if* $x \in \{0,1\}$ *or* $y \in \{0,1\}$.

11. $e_c(x,\nu_c) \geq \nu_c$ *and similarly,* $e_d(x,\nu_d) \geq \nu_d$.

Proof. 1. From $f_c^{-1}(0) = 1$, it follows that $e_c(1,1) = e_c(0,0) = 1$. From $f_c(1) = 0$, $f_c(0) = 1$ and $f_c^{-1}(1) = 0$, we get that $e_c(0,1) = e_c(1,0) = 0$. Similarly, from $f_d^{-1}(1) = 1$, it follows that $e_d(1,1) = e_d(0,0) = 1$. From $f_d(1) = 1$, $f_d(0) = 0$ and $f_d^{-1}(0) = 0$, we get that $e_d(0,1) = e_d(1,0) = 0$.

2. Trivial.

3. $e_c(x,x) = f_c^{-1}(0) = 1$ and $e_d(x,x) = f_d^{-1}(1) = 1$.

4. We have to show that from $x \leq x' \leq y' \leq y$ it follows that $e_c(x,y) \leq e_c(x',y')$. Using the monotonicity of $f_c(x)$ and $f_c^{-1}(x)$, the statement follows immediately. For e_d, we have to show that from $x \leq x' \leq y' \leq y$ it follows that $e_d(x,y) \leq e_d(x',y')$. Using the monotonicity of $f_d(x)$ and $f_d^{-1}(x)$ the statement follows immediately.

5. By using the decreasing property of f_c^{-1} and the triangle inequality, we obtain

$$c(e(x,y),e(y,z)) = f_c^{-1}(|f_c(x)-f_c(y)|+|f_c(y)-f_c(z)|) \leq f_c^{-1}(|f_c(x)-f_c(z)|) = e(x,z).$$

The proof is similar for e_d as well.

6. $e_c(x,y) \geq \nu_c$ iff $|f_c(x)-f_c(y)| \leq \frac{1}{2}$ and similarly, $e_c(y,z) \geq \nu_c$ iff $|f_c(y)-f_c(z)| \leq \frac{1}{2}$. Obviously, these conditions are not sufficient for $|f_c(x) - f_c(z)| \leq \frac{1}{2}$. Similarly, $e_d(x,y) \geq \nu_d$ iff $1 - |f_d(x) - f_d(y)| \geq \frac{1}{2}$ and similarly, $e_d(y,z) \geq \nu_d$ iff $|f_d(y) - f_d(z)| \geq \frac{1}{2}$. Obviously, these conditions are not sufficient for $1-|f_d(x)-f_d(z)| \geq \frac{1}{2}$.

7. $e_c(n_c(x),n_c(y)) = f_c^{-1}[|f_c(n_c(x)) - f_c(n_c(y))|] = f_c^{-1}[|1 - f_c(x) - (1 - f_c(y))|] =$

$f_c^{-1}[|f_c(y) - f_c(x)|] = e_c(x,y)$. Similarly, $e_d(n_d(x),n_d(y)) = f_d^{-1}[|f_d(n_d(x)) - f_d(n_d(y))|] =$

$f_d^{-1}[|1 - f_d(x) - (1 - f_d(y))|] = f_d^{-1}[|f_d(y) - f_d(x)|] = e_d(x,y)$.

8. Using the fact that $f_c(1) = 0$, we get $e_c(1,x) = f_c^{-1}[|f_c(1) - f_c(x)|] = x$.

Similarly, using the fact that $f_c(0) = 1$ and that $0 \leq f_c(x) \leq 1$ for $\forall x \in [0,1]$, we get $e_c(0,x) = f_c^{-1}[|f_c(0) - f_c(x)|] = n_c(x)$. For e_d, using the fact that $f_d(1) = 1$ and that $0 \leq f_d(x) \leq 1$ for $\forall x \in [0,1]$ we get $e_d(1,x) = f_d^{-1}[1 - |f_d(1) - f_d(x)|] = x$. From $f_d(0) = 0$, we get $e_d(0,x) = f_d^{-1}[1 - |f_d(0) - f_d(x)|] = n_d(x)$.

9. If $e_c(x,y) = 0$, then $|f_c(x) - f_c(y)| = 1$, from which $x,y \in 0,1$ and $x \neq y$. Going in the opposite direction is trivial.

10. $n_c(e_c(x,y)) = f_c^{-1}(1-|f_c(x)-f_c(y)|)$ and $e_c(n_c(x),y)) = f_c^{-1}(|1-f_c(x)-f_c(y)|)$. Considering the four cases and using the monotonicity of $f_c(x)$, we get that $x \in \{0,1\}$ or $y \in \{0,1\}$. The proof is similar for $e_d(x,y)$ as well.

11. Using the monotonicity property of $f_c(x)$ and the fact that $f_c(\nu_c) = \frac{1}{2}$, we get $e_c(x,\nu_c) = f_c^{-1}\left[|f_c(x)-f_c(\nu_c)|\right] = f_c^{-1}\left[|f_c(x)-\frac{1}{2}|\right] \geq \nu_c$, since $0 \leq \left[|f_c(x)-\frac{1}{2}|\right] \leq \frac{1}{2}$. Similarly, using the monotonicity property of $f_d(x)$ and the fact that $f_d(\nu_d) = \frac{1}{2}$, we get

$$e_d(x,\nu_d) = f_d^{-1}\left[1-|f_d(x)-f_d(\nu_d)|\right] = f_d^{-1}\left[1-|f_d(x)-\frac{1}{2}|\right] \geq \nu_d,$$

since $\frac{1}{2} \leq 1-|f_d(x)-\frac{1}{2}| \leq 1$.

□

Proposition 6.11. *If $x,y > \nu_c$ or $x,y < \nu_c$, then $e_c(x,y) > \nu_c$. Similarly, if $x,y > \nu_d$ or $x,y < \nu_d$, then $e_d(x,y) > \nu_d$.*

Proof. If $x,y > \nu_c$, then $f_c(x), f_c(y) < \frac{1}{2}$, so $|f_c(x)-f_c(y)| < \frac{1}{2}$, which means that $e_c(x,y) > \nu_c$. Similarly, if $x,y < \nu_c$, then $f_c(x), f_c(y) > \frac{1}{2}$, so $|f_c(x)-f_c(y)| < \frac{1}{2}$, which means that $e_c(x,y) > \nu_c$. For e_d, if $x,y > \nu_d$, then $f_d(x), f_d(y) > \frac{1}{2}$, so $|f_d(x)-f_d(y)| < \frac{1}{2}$, which means that $e_d(x,y) > \nu_d$. Similarly, if $x,y < \nu_d$, then $f_d(x), f_d(y) < \frac{1}{2}$, so $|f_d(x)-f_d(y)| < \frac{1}{2}$, which means that $e_d(x,y) > \nu_d$.

□

Remark 6.12. e_c and e_d are not associative.

Proof. A possible counterexample might be the case of rational generators with $\nu_c = 0.6$ and $\nu_d = 0.3$, $x = 0.3$, $y = 0.4$ and $y = 0.5$. In this case we get $e_c(x, e_c(y,z)) \approx 0.39$, $e_c(e_c(x,y),z) \approx 0.62$, while for $e_d(x,e_d(y,z)) \approx 0.38$ and $e_d(e_d(x,y),z) \approx 0.64$.

□

Proposition 6.13. *In a connective system the above-defined equivalences $e_c(x,y)$ and $e_d(x,y)$ coincide if and only if $f_c(x) + f_d(x) = 1$ (or equivalently $n_c = n_d$, i.e. in a Lukasiewicz system), where f_c and f_d are the normalized generation function of the conjunction and disjunction operators, respectively.*

Proof. 1. If $f_c(x) + f_d(x) = 1$, then $f_c(x) = 1 - f_d(x)$ and $f_c^{-1}(x) = f_d^{-1}(1-x)$, from which we get $e_c(x,y) = f_c^{-1}\left[|f_c(x)-f_c(y)|\right] = f_d^{-1}\left[1-|f_d(x)-f_d(y)|\right] = e_d(x,y)$.

2. If $e_c(x, y) = e_d(x, y)$, then in particular $e_c(0, x) = e_d(x, 0)$, which means that $n_c(x) = n_d(x)$ must hold for all $x \in [0, 1]$.

$\square$

6.4 Dual equivalences

In classical logic, the equivalence operator has the following important property as well: $e(x, n(x)) = 0$. As it is well known, demanding $e(x, x) = 1$ and $e(x, n(x)) = 0$ at the same time gives rise to a paradox.

Lemma 6.14. *There is no equivalence relation which fulfils both $e(x, x) = 1$ and $e(x, n(x)) = 0$.*

Proof. Let ν be the fix point of the negation $n(x)$. Then $1 = e(\nu, \nu) = e(\nu, n(\nu)) = 0$, which is a contradiction. $\square$

However, in practical applications the property $e(x, n(x)) = 0$ might be of even greater importance than reflexivity (see Dombi, [26]). Motivated by this demand, I define new types of operators below.

First, the so-called dual equivalence is defined, denoted by $\bar{e}$. Let us now consider a nilpotent connective system (c, d, n) and let us denote the normalized generator functions of c and d by f_c and f_d, respectively.

Definition 6.15. The dual equivalence operations are defined as follows.

$\bar{e}_c(x, y) = n_c\left(e_c(x, n_c(y)\right)$ and

$\bar{e}_d(x, y) = n_d\left(e_d(x, n_d(y)\right).$

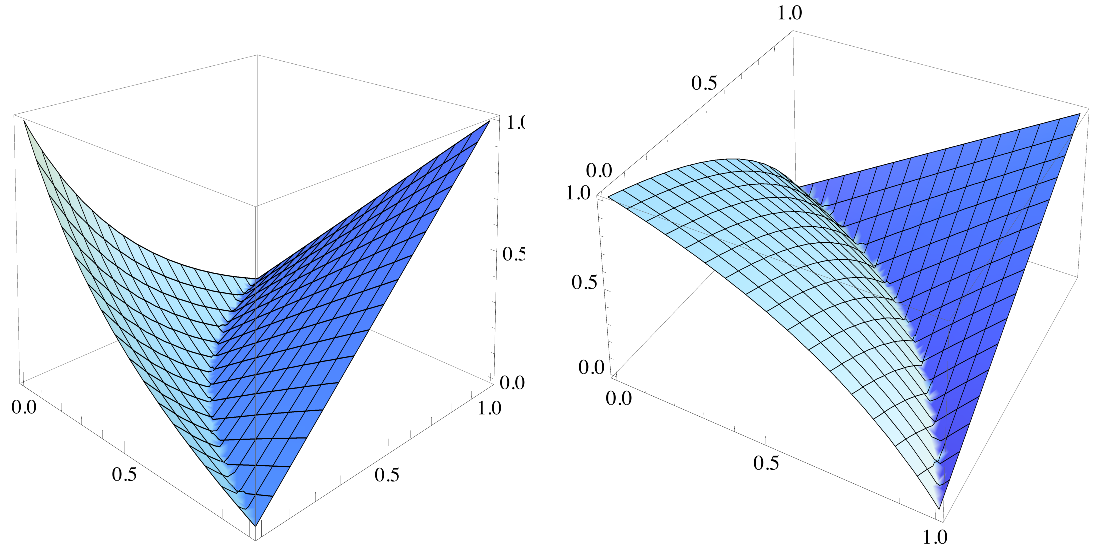
Proposition 6.16. *In a bounded system the equivalence operators have the form*

$\bar{e}_c(x, y) = f_c^{-1}\left[1 - |f_c(x) + f_c(y) - 1|\right]$ *and*

$\bar{e}_d(x, y) = f_d^{-1}\left[|f_d(x) + f_d(y) - 1|\right].$

Proof. The formulae can be derived from direct calculation. $\square$

Remark 6.17. Since $0 \leq |f_c(x) + f_c(y) - 1| \leq 1$ and $0 \leq |f_d(x) + f_d(y) - 1| \leq 1$, the cutting function can be omitted here. For conceptual reasons, I prefer to leave it in all of the formulae.



(a) $\bar{e}_c$ with $\nu_c = 0.6$ (b) $\bar{e}_d$ with $\nu_d = 0.6$

FIGURE 6.2: $\bar{e}_c(x, y)$ and $\bar{e}_d$ with rational generators

6.4.1 Properties of $\bar{e}_d$ and $\bar{e}_c$

Next, the main properties of the dual equivalences are studied.

Proposition 6.18. *Let ν_c and ν_d, be the fixpoints of n_c and n_d, respectively. Then the operators $\bar{e}_c(x, y)$ and $\bar{e}_d(x, y)$ have the following properties:*

1. *Compatibility (see Definition 6.5).*

2. *Symmetry (see Definition 6.5).*

3. *$\bar{e}_c(x, y)$ and $\bar{e}_d(x, y)$ are not reflexive, but $\bar{e}_c(x, n_c(x)) = \bar{e}_d(x, n_d(x)) = 0$.*

4. *$\bar{e}_c(x, y)$ and $\bar{e}_d(x, y)$ are not monotonic.*

5. *$\bar{e}_c$ is T-transitive with respect to the conjunction c (see Definition 6.6) and similarly, $\bar{e}_d$ is T-transitive with respect to the t-norm generated by $1 - f_d(x)$.*

6. *$\bar{e}_c(x, y)$ and $\bar{e}_d(x, y)$ are not threshold transitive with respect to ν_c and ν_d (see Definition 6.6).*

7. *Invariance with respect to n_c and n_d (see Definition 6.6).*

8. *$\bar{e}_c(1, x) = \bar{e}_c(1, x) = x$*

   $\bar{e}_d(0, x) = n_d(x)$, and similarly, $\bar{e}_c(0, x) = n_c(x)$.

9. *$\bar{e}_c(x, y) = 0$ if and only if $x = n_c(y)$ and similarly, $\bar{e}_d(x, y) = 0$ if and obly if $x = n_d(y)$.*

10. $n_d(\bar{e}_d(x,y)) = \bar{e}_d(n_d(x),y))$ *if and only if* $x \in \{0,1\}$ *or* $y \in \{0,1\}$ *and* $n_c(\bar{e}_c(x,y)) = \bar{e}_c(n_c(x),y))$ *if and only if* $x \in \{0,1\}$ *or* $y \in \{0,1\}$.

11. $\bar{e}_c(x,\nu_c) \leq \nu_c$ *and* $\bar{e}_d(x,\nu_d) \leq \nu_d$.

Proof. 1. Using the formulae given in Proposition 6.16, compatibility is trivial.

2. Using the formulae given in Proposition 6.16, symmetry is trivial as well.

3. Follows from direct calculation. Since $\bar{e}_c(x,n_c(x)) = 0$ holds for the fixpoint ν_c of the n_c as well, reflexivity cannot hold. Similarly for $\bar{e}_d$.

4. A counterexample might be the case of rational generators with $\nu_c = 0.3$. $\bar{e}_c(0.1,0.6) \approx 0.75$, while $\bar{e}_c(0.4,0.5) \approx 0.68$, and similarly for $\bar{e}_d(0.4,0.6) \approx 0.21$, while $\bar{e}_d(0.45,0.5) \approx 0.19$.

5. By using the decreasing property of f_c^{-1} and the fact that $|a+b-1|+|b+c-1|-1 \leq |a+c-1|$ holds for all $a,b,c \in [0,1]$, we obtain

$$c(\bar{e}_c(x,y),\bar{e}_c(y,z)) = f_c^{-1}(2 - |f_c(x)+f_c(y)-1| - |f_c(y)+f_c(z)-1|) \leq$$

$$\leq f_c^{-1}(1 - |f_c(x)+f_c(z)-1|) = \bar{e}_c(x,z).$$

The proof is similar for $\bar{e}_d$ as well.

6. A possible counterexample might be for rational generators with $\nu_c = 0.3$, $x = 0.85, y = 0.9$ and $z = 0.87$, or for $\nu_d = 0.3$, $x = 0.7, y = 0.9$ and $z = 0.6$.

7. $\bar{e}_c(n_c(x),n_c(y)) = 1 - f_c^{-1}\left[|1-f_c(x)+1-f_c(y)-1|\right] = \bar{e}_c(x,y)$ and similarly, $\bar{e}_d(n_d(x),n_d(y)) = f_d^{-1}\left[|1-f_d(x)+1-f_d(y)-1|\right] = \bar{e}_d(x,y)$.

8. Using the fact that $f_c(1) = 0$, we get $\bar{e}_c(1,x) = f_c^{-1}\left[1-|f_c(1)+f_c(x)-1|\right] = x$.

   Similarly, using the fact that $f_c(0) = 1$ and that $0 \leq f_c(x) \leq 1$ for $\forall x \in [0,1]$, we get $\bar{e}_c(0,x) = f_c^{-1}\left[1-|f_c(0)+f_c(x)-1|\right] = n_c(x)$. For e_d, using the fact that $f_d(1) = 1$ and that $0 \leq f_d(x) \leq 1$ for $\forall x \in [0,1]$ we get $\bar{e}_d(1,x) = f_d^{-1}\left[|f_d(1)+f_d(x)-1|\right] = x$. Using the fact that $f_d(0) = 0$, we get $\bar{e}_d(0,x) = f_d^{-1}\left[|f_d(0)-f_d(x)-1|\right] = n_d(x)$.

9. Using the fact that $f_c(n_c(x)) = 1 - f_c(x)$ and $f_d(n_d(x)) = 1 - f_d(x)$, we get $\bar{e}_c(x,n_c(x)) = 1 - f_c^{-1}(0) = 0$ and similarly $\bar{e}_d(x,n_d(x)) = f_d^{-1}(0) = 0$. If $\bar{e}_c(x,y) = 0$, then $f_c(x)+f_c(y) = 1$, from which $f_c(x) = 1 - f_c(y)$, i.e. $x = f_c^{-1}[1-f_c(y)] = n_c(y)$. Similarly, if $\bar{e}_d(x,y) = 0$, then $f_d(x)+f_d(y) = 1$, from which $f_d(x) = 1 - f_d(y)$, i.e. $x = f_d^{-1}[1-f_d(y)] = n_d(y)$.

10. $n_c(\bar{e}_c(x,y)) = f_c^{-1}(1-|f_c(x)+f_c(y)-1|)$ and $\bar{e}_c(n_c(x),y)) = f_c^{-1}(1-|f_c(x)-f_c(y)|)$. Considering the four cases and using the monotonicity of $f_c(x)$, we get that $x \in \{0,1\}$ or $y \in \{0,1\}$. The proof for $e_d(x,y)$ follows in a similar way.

11. Using the strict monotonicity of f_c, f_d and their inverse functions, and the fact that $f_c(\nu_c) = f_d(\nu_d) = \frac{1}{2}$, the proof can be found by direct calculation.

□

Remark 6.19. $\bar{e}_c(x,y)$ and $\bar{e}_d(x,y)$ are not associative.

Proof. It is easy to find a counterexample, e.g. for rational generators with $\nu_c = 0.3$, $\bar{e}_c(0.3, \bar{e}_c(0.4, 0.5)) \approx 0.58$, while $\bar{e}_c(\bar{e}_c(0.3, 0.4), 0.5) \approx 0.16$.

Similarly, $\bar{e}_d(0.1, \bar{e}_d(0.5, 0.7)) \approx 0.12$, while $\bar{e}_d(\bar{e}_c(0.1, 0.5), 0.7) \approx 0.03$. □

Proposition 6.20. *In a connective system the above-defined equivalences $\bar{e}_c(x,y)$ and $\bar{e}_d(x,y)$ coincide if and only if $f_c(x) + f_d(x) = 1$ (or equivalently $n_c = n_d$, i.e. in a Łukasiewicz system), where f_c and f_d are the normalized generation function of the conjunction and disjunction operators, respectively.*

Proof.

1. If $f_c(x) + f_d(x) = 1$, then $f_c(x) = 1 - f_d(x)$ and $f_c^{-1}(x) = f_d^{-1}(1-x)$, from which we get $\bar{e}_c(x,y) = f_c^{-1}\left[1 - |f_c(x) + f_c(y) - 1|\right] = f_d^{-1}\left[|1 - f_d(x) - f_d(y)|\right] = e_d(x,y)$.

2. If $e_c(x,y) = e_d(x,y)$, then in particular $\bar{e}_c(0,x) = \bar{e}_d(x,0)$, which means that $n_c(x) = n_d(x)$ must hold for all $x \in [0,1]$.

□

6.5 Arithmetic mean operators in bounded systems

Let us define the so-called arithmetic mean operators in a bounded system.

Definition 6.21. In a connective system (c, d, n)

$$m_c^{(\alpha)}(x,y) := f_c^{-1}\left[\alpha \cdot f_c(x) + (1-\alpha) \cdot f_c(y)\right]$$

and similarly,

$$m_d^{(\alpha)}(x,y) := f_d^{-1}\left[\alpha \cdot f_d(x) + (1-\alpha) \cdot f_d(y)\right],$$

where f_c and f_d are the normalized generator functions of the conjunction and disjunction operators, respectively, $0 < \alpha < 1$. m_c and m_d are called weighted arithmetic mean operators.

Proposition 6.22. *$m_c^{(\alpha)}(x, y)$ and $m_d^{(\alpha)}(x, y)$ satisfy the self-De Morgan property with respect to n_c and n_d respectively, i.e.*

$$n_c\left(m_c^{(\alpha)}(x,y)\right) = m_c^{(\alpha)}\left(n_c(x), n_c(y)\right)$$

and similarly,

$$n_d\left(m_d^{(\alpha),}(x,y)\right) = m_d^{(\alpha),}\left(n_d(x), n_d(y)\right).$$

Proof.

$$n_c\left(m_c^{(\alpha)}(x,y)\right) = f_c^{-1}\left[1 - (\alpha \cdot f_c(x) + (1-\alpha) \cdot f_c(y))\right] =$$

$$= f_c^{-1}\left[\alpha \cdot (1 - f_c(x)) + (1-\alpha) \cdot (1 - f_c(y)))\right] = m_c^{(\alpha)}\left(n_c(x), n_c(y)\right).$$

For m_d, the proof is similar. □

6.6 Aggregated equivalences

Next, I define a new type of operator derived from the equivalences defined above. This new operator is a compromise between the normal and the dual equivalences, i.e. it fulfils neither $e(x, x) = 1$ nor $e(x, n(x)) = 0$, but it has a nice property, namely $e(\nu, \nu) = \nu$. If we recall that the values represent uncertainities and ν, as the fix point of the negation means that we hesitate whether the objects A and B have the particular property or not, it is also sensible to remain unsure about their equivalence value. This new operator will be called the aggregated equivalence operator.

Definition 6.23. The aggregated equivalence operators are defined as follows.

$$e_c^*(x,y) = m_c^{\frac{1}{2}}\left(e_c(x,y), \bar{e}_c(x,y)\right),$$

$$e_d^*(x,y) = m_d^{\frac{1}{2}}\left(e_d(x,y), \bar{e}_d(x,y)\right).$$

Proposition 6.24. *The aggregated equivalence operator in a bounded system*

$$e_c^*(x,y) = f_c^{-1}\left[\frac{1}{2}|f_c(x) - f_c(y)| + \frac{1}{2}\left(1 - |f_c(x) + f_c(y) - 1|\right)\right]$$

and

$$e_d^*(x,y) = f_d^{-1}\left[\frac{1}{2}\left(1 - |f_d(x) - f_d(y)|\right) + \frac{1}{2}|f_d(x) + f_d(y) - 1|\right].$$

Proof. Follows from direct calculation. □

Proposition 6.25. *The conjunctive aggregated equivalence operator has the following property.*

$$e_c^*(x,y) = \begin{cases} n_c(y), & \text{if } x \le y \le n_c(x) \\ x, & \text{if } n_c(y) \le x \le y \\ n_c(x), & \text{if } y \le x \text{ and } y \le n_c(x) \\ y, & \text{if } y \le x \text{ and } y \ge n_c(x). \end{cases}$$

Proof.

1. If $x \le y \le n_c(x)$, then using the monotonicity of f_c and the fact that $n_c(x) = f_c^{-1}(1 - f_c(x))$, we get $f_c(x) \ge f_c(y)$ and $f_c(x) + f_c(y) \ge 1$. In this case it means that $e_c^*(x,y) = n(y)$.

2. If $n_c(y) \le x \le y$, then using the monotonicity of f_c and the fact that $n_c(x) = f_c^{-1}(1 - f_c(x))$ we get $f_c(x) \ge f_c(y)$ and $f_c(x) + f_c(y) \le 1$. In this case it means that $e_c^*(x,y) = x$.

3. If $y \le x$ and $y \le n_c(x)$, then we get $f_c(x) \le f_c(y)$ and $f_c(x) + f_c(y) \ge 1$. In this case $e_c^*(x,y) = n_c(x)$ follows.

4. If $y \le x$ and $y \ge n_c(x)$, then $f_c(x) \le f_c(y)$ and $f_c(x) + f_c(y) \le 1$. In this case it means that $e_c^*(x,y) = y$.

□

Proposition 6.26. *The disjunctive aggregated equivalence operator has the following property.*

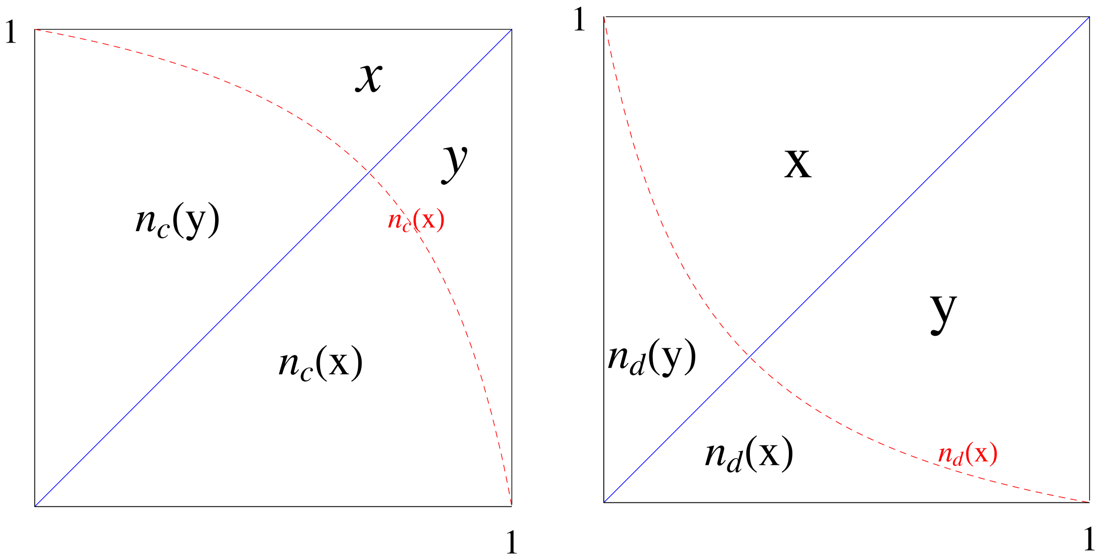
$$e_d^*(x,y) = \begin{cases} n_d(y), & \text{if } x \le y \text{ and } x \le n_d(y) \\ x, & \text{if } n_d(y) \le x \le y \\ n_d(x), & \text{if } y \le x \text{ and } x \le n_d(y) \\ y, & \text{if } y \le x \text{ and } n_d(y) \le x. \end{cases}$$

Proof.

1. If $x \le y$ and $x \le n_d(y)$, then using the monotonicity of f_d and the fact that $n_d(x) = f_d^{-1}(1 - f_d(x))$ we get $f_d(x) \le f_d(y)$ and $f_d(x) + f_d(y) \le 1$. In this case it follows that $e_d^*(x,y) = n_d(y)$.

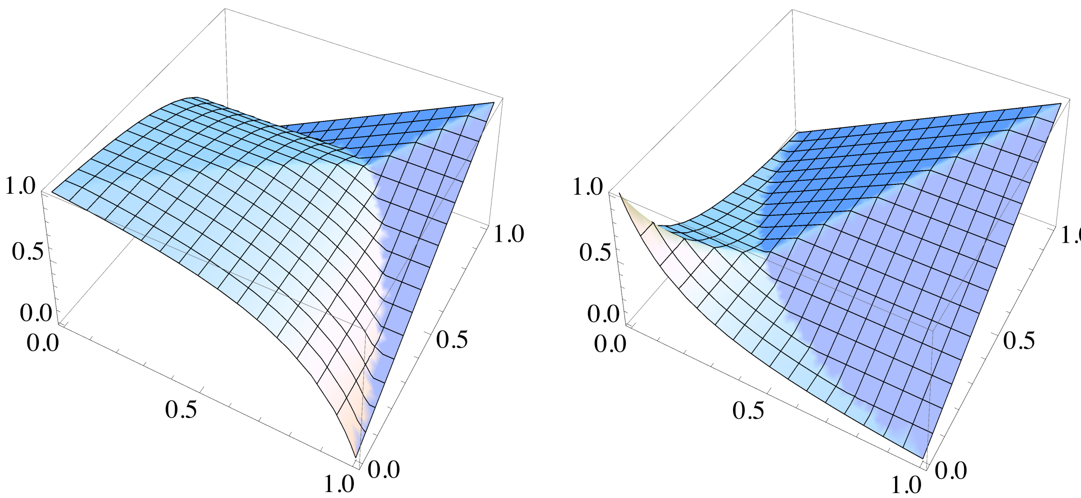
2. If $n_d(y) \le x \le y$, then we get $f_d(x) \le f_d(y)$ and $f_d(x) + f_d(y) \ge 1$. In this case it follows that $e_d^*(x,y) = x$.

3. If $y \le x$ and $x \le n_d(y)$, then we get $f_d(x) \ge f_d(y)$ and $f_d(x) + f_d(y) \le 1$. In this case it follows that $e_d^*(x,y) = n_d(x)$.



(a) e_c^* (b) e_d^*

FIGURE 6.3: The domain of aggregated equivalences



(a) e_c^* (b) e_d^*

FIGURE 6.4: Aggregated equivalences with rational generators with $\nu = 0.3$

4. If $y \le x$ and $n_d(y) \le x$, then $f_d(x) \ge f_d(y)$ and $f_d(x) + f_d(y) \ge 1$. In this case it follows that $e_d^*(x, y) = y$.

□

6.6.1 Properties of the aggregated equivalence operator

Next, the main properties of the aggregated equivalences are examined. In Propositions 6.27 and 6.31, I will show that unlike the above-mentioned equivalences, the aggregated equivalences are threshold transitive and associative as well.

Proposition 6.27. *Let ν_c and ν_d be the fixpoints of n_c and n_d, respectively. The aggregated equivalences have the following properties:*

1. *Compatibility (see Definition 6.5).*

2. *Symmetry (see Definition 6.5).*

3. *The aggregated equivalences are not reflexive, but $e^*_c(\nu_c, \nu_c) = \nu_c$ and $e^*_d(\nu_d, \nu_d) = \nu_d$ hold. In addition,*
$$e^*_c(x, x) = \begin{cases} n_c(x), & \text{if } x \leq \nu_c \\ x, & \text{if } x \geq \nu_c. \end{cases}$$
and similarly,
$$e^*_d(x, x) = \begin{cases} n_d(x), & \text{if } x \leq \nu_d \\ x, & \text{if } x \geq \nu_d. \end{cases}$$

4. *Monotonicity (see Definition 6.5).*

5. *e^*_c is T-transitive with respect to the conjunction c (see Definition 6.6) and similarly, e^*_d is T-transitive with respect to the t-norm generated by $1 - f_d(x)$.*

6. *The aggregated equivalences are threshold transitive with respect to ν_c and ν_d (see Definition 6.6).*

7. *Invariance with respect to n_c and n_d (see Definition 6.6).*

8. *$e^*_c(1, x) = e^*_d(1, x) = x$, $e^*_d(0, x) = n_d(x)$, and similarly, $e^*_c(0, x) = n_c(x)$.*

9. *$e^*_c(x, y) = 0$ if and only if $x, y \in 0, 1$ and $x \neq y$. Similarly for e^*_d.*

10. *$n_c(e^*_c(x, y)) = e^*_c(n_c(x), y))$ if and only if $x \in \{0, 1\}$ or $y \in \{0, 1\}$ and $n_d(e^*_d(x, y)) = e^*_d(n_d(x), y))$ if and only if $x \in \{0, 1\}$ or $y \in \{0, 1\}$.*

11. *$e^*_c(x, \nu_c) = \nu_c$ and similarly, $e^*_d(x, \nu_d) = \nu_d$.*

Proof. 1. Follows from direct calculation.

2. Trivial.

3. The statement follows from Proposition 6.25 and 6.26.

4. I show monotonicity for e^*_c. For e^*_d the proof is similar. If $x \leq x' \leq y' \leq y$, then by Proposition 6.25 we have to consider two cases.

   (a) $y \leq n_c(x)$. In this case $e^*_c(x, y) = n_c(y)$.

i. If $y' \leq n_c(x')$, then $e^*_c(x', y') = n_c(y')$, which means that $e^*_c(x, y) \leq e^*_c(x', y')$.

ii. If $y' \geq n_c(x')$, then $e^*_c(x', y') = x'$ and $n_c(y) \leq n_c(y') \leq x'$, so $e^*_c(x, y) \leq e^*_c(x', y')$.

(b) $y \geq n_c(x)$. In this case $e^*_c(x, y) = x$.

i. If $y' \geq n_c(x')$, then $e^*_c(x', y') = x'$, which means that $e^*_c(x, y) \leq e^*_c(x', y')$.

ii. If $y' \leq n_c(x')$, then $e^*_c(x', y') = n_c(y')$ and $n_c(y') \geq x' \geq x$, so $e^*_c(x, y) \leq e^*_c(x', y')$.

5. By using the decreasing property of f_c^{-1} and the fact that $|a-b|-|a+b-1|+|b-c|-|b+c-1|+1 \geq |a-c|-|a+c-1|$ holds for all $a, b, c \in [0, 1]$, the statement follows from direct calculation. The proof is similar for e^*_d as well.

6. I show the threshold transitivity for e^*_c. For e^*_d, the proof is similar.

   The condition $e^*_c(x, y) \geq \nu_c$ is equivalent to the following inequality.

$$f_c^{-1}\left[\frac{1}{2}|f_c(x) - f_c(y)| + \frac{1}{2}\left(1 - |f_c(x) + f_c(y) - 1|\right)\right] \geq \nu_c,$$

   which means that

$$|f_c(x) - f_c(y)| \leq |f_c(x) + f_c(y) - 1|.$$

   This means that either $f_c(x), f_c(y) \leq \frac{1}{2}$, or $f_c(x), f_c(y) \geq \frac{1}{2}$ must hold, i.e. either $x, y \geq \nu_c$, or $x, y \leq \nu_c$.

   Together with the condition $e^*_c(y, z) \geq \nu_c$, we also have that $y, z \geq \nu_c$, or $y, z \leq \nu_c$, from which we easily get that either $x, z \geq \nu_c$, or $x, z \leq \nu_c$ must hold, i.e. $e^*_c(x, z) \geq \nu_c$.

7. Follows from direct calculation.

8. Follows from the properties of e_c, $\bar{e_c}$, e_d, and $\bar{e_d}$.

9. The statement follows from propositions 6.25 and 6.26.

10. Follows from direct calculation.

11. $e^*_c(x, \nu_c) = f_c^{-1}\left[\frac{1}{2}|f_c(x) - \frac{1}{2}| + \frac{1}{2}\left(1 - |f_c(x) - \frac{1}{2}|\right)\right] = f_c^{-1}\left(\frac{1}{2}\right) = \nu_c$. Similarly for e^*_d as well.

□

Remark 6.28. Note that from Proposition 3, it follows immediately that $e^*_c(x, x) \geq \nu_c$ and similarly for e^*_d as well.

Proposition 6.29. *$e_c^*(x,y) > \nu_c$ if and only if $x, y > \nu_c$ or $x, y < \nu_c$, $e_c^*(x,y) = \nu_c$ if and only if $x = \nu_c$ or $y = \nu_c$, and $e_c^*(x,y) < \nu_c$ otherwise. Similarly for $e_d^*(x,y)$.*

Proof. The statement readily follows from propositions 6.25 and 6.26.

□

Remark 6.30. Note that e_c^* and e_d^* considered as fuzzy binary relations on [0,1], are both c-transitive (see page 53, Fodor and Roubens, [38]).

Proposition 6.31. *e_c^* and e_d^* are associative.*

Proof. Let us consider $e_d^*(x,y)$. First, I will show that associativity holds in the case where $f_d(x) = 1 - x$. Let us use the following notation for the disjunctive aggregated equivalence for $f_d(x) = 1 - x$.

$$L(x,y) := e_d^*(x,y) = \frac{1}{2}\left(|x+y-1| - |x-y| + 1\right).$$

It can be shown that

$$L(x,y) = \min(\max(1-x,y), \max(x,1-y)).$$

From this, we get

$$L(x,L(y,z)) = \min(\max(x,y,z), \max(x,1-y,1-z), \max(1-x,y,1-z), \max(1-x,1-y,z)) =$$

$$= L(L(x,y),z),$$

which means that $L(x,y)$ is associative. In particular, for an arbitrary generator function f_d,

$$f_d^{-1}\left(L(f_d(x), L(f_d(y), f_d(z)))\right) = f_d^{-1}\left(L(L(f_d(x), f_d(y)), f_d(z))\right)$$

also holds. Since

$$e_d^*(x,y) = f_d^{-1}\left[\frac{1}{2}\left(1 - |f_d(x) - f_d(y)|\right) + \frac{1}{2}|f_d(x) + f_d(y) - 1|\right] = f_d^{-1}\left(L(f_d(x), f_d(y))\right),$$

associativity of $e_d^*(x,y)$ is proved. The proof for e_c^* is similar as well.

□

Proposition 6.32. *In a connective system, the above-defined equivalences $e_c^*(x,y)$ and $e_d^*(x,y)$ coincide if and only if $f_c(x) + f_d(x) = 1$ (or equivalently $n_c = n_d$, i.e. in a Łukasiewicz system), where f_c and f_d are the normalized generation function of the conjunction and disjunction operators, respectively.*

Proof. 1. If $f_c(x)+f_d(x)=1$, then using the fact that $f_c(x)=1-f_d(x)$ and $f_c^{-1}(x)=f_d^{-1}(1-x)$, we get $e_c^*(x,y)=e_d^*(x,y)$.

2. If $e_c^*(x,y)=e_d^*(x,y)$, then in particular $e_c^*(0,x)=e_d^*(x,0)$, which means that $n_c(x)=n_d(x)$ must hold for all $x\in[0,1]$.

□

6.7 Applications

In signal and image processing, the equivalence of two signals or two images is always of great importance.

Let us assume that two grey level images, i.e. two integer-valued function f and g defined on a subinterval I of $\mathbb{Z}^2$, are given. After normalizing f and g, the equivalence of the images can be calculated in each picture element x of I (pixel) by using the equivalence operators considered above. For simplicity, let us assume that $I=\{0,...,n\}^2$, and let us use the following notations: $x_{i,j}:=f(i,j)$ and $y_{i,j}:=g(i,j)$. The overall equivalence of the two images (which measures the overlap) can be calculated by an arithmetic mean in the following way.

Definition 6.33. Let us consider two normalized grey level images, $f,g:I\to[0,1]$, where $I=\{0,...,n\}^2$. Their overall equivalence E is defined the following way:

$$E(f,g):=\frac{1}{n^2}\sum_{i,j=1}^{n}e(x_{i,j},y_{i,j}),$$

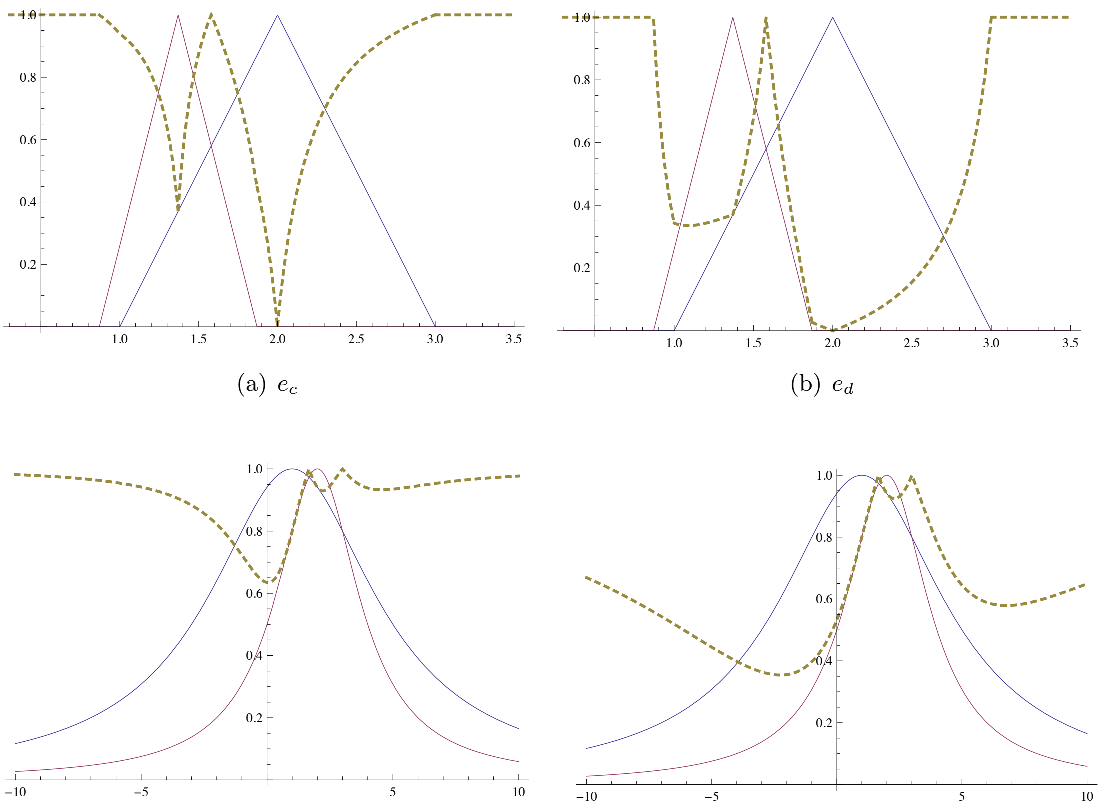
where $x_{i,j}=f(i,j)$ and $y_{i,j}=g(i,j)$, and e stands for one of the equivalences considered so far.

The overall equivalence can be defined for one dimensional signals similarly.

Note that for values around the middle grey level, the aggregated equivalences, e_c^* and e_d^*, give the maximal level of uncertainty, which fact gives them an important semantic meaning. Therefore, when studying the equivalence of two grey level images, the aggregated equivalences are of great importance.

6.8 Overview

The main properties of all the three types of the above mentioned equivalence operators are summarized in Table A.8.



(a) e_c (b) e_d

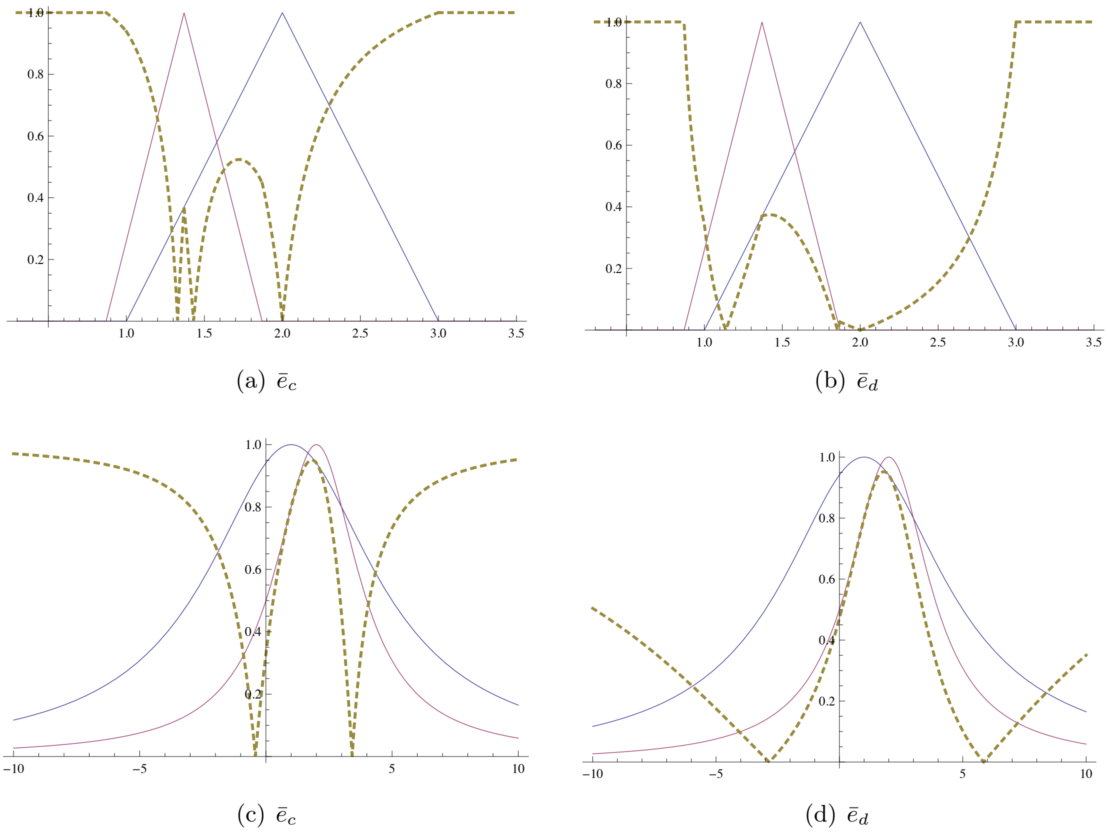
(c) e_c (d) e_d

FIGURE 6.5: Pointwise equivalence of fuzzy numbers with rational generators ($\nu_c = \nu_d = 0.3$)

In figures 6.5, 6.6 and 6.7, examples of the pointwise equivalence of two fuzzy numbers are illustrated by means of all the above mentioned equivalences.

Thesis 2.4.

A detailed discussion of equivalence operators in bounded systems are given. Three different types of operators are studied. After taking a closer look at the implication-based equivalences, the properties of the so-called dual equivalences are studied. Using these two types of equivalence operators, a new concept of aggregated equivalences is introduced. The paradox of the equivalence relation is solved by aggregating the implication-based equivalence and its dual operator. It is shown that the aggregated equivalence possesses nice properties like threshold transitivity, T-transitivity and associativity. For applications in image processing, the overall equivalence of two grey level images was defined, and an important semantic meaning of the aggregated equivalences is given.



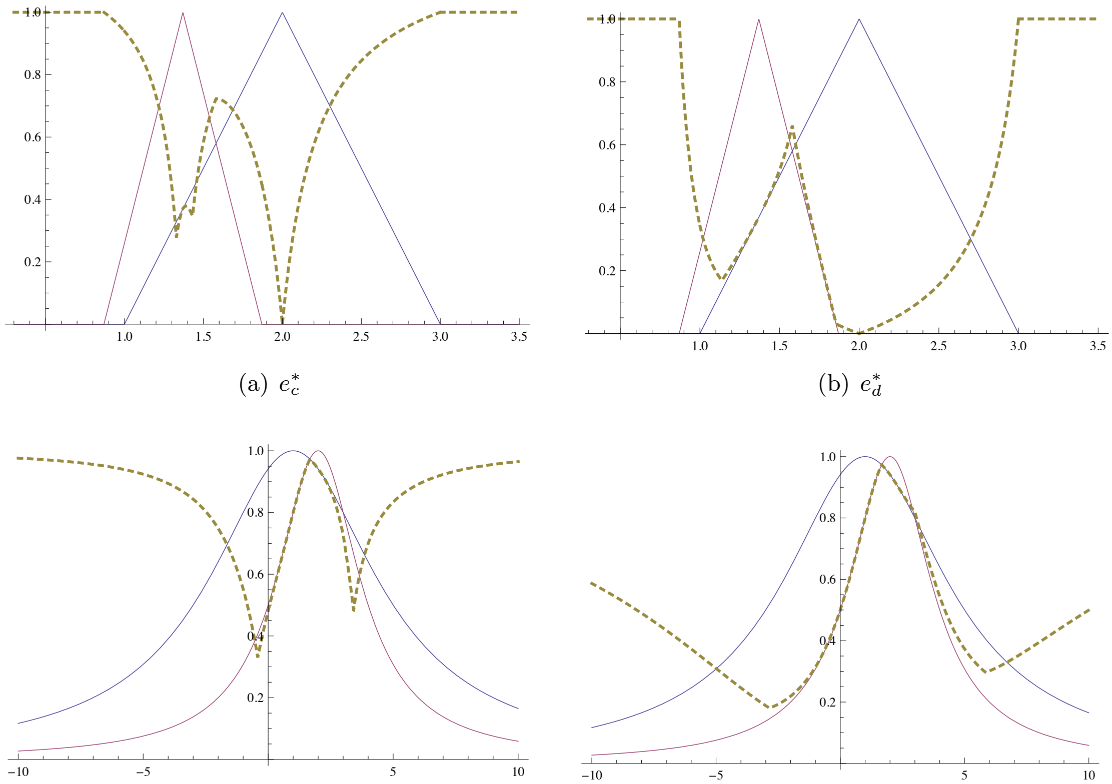
(a) $\bar{e}_c$

(b) $\bar{e}_d$

(c) $\bar{e}_c$

(d) $\bar{e}_d$

FIGURE 6.6: Pointwise dual equivalence of fuzzy numbers with rational generators ($\nu_c = \nu_d = 0.3$)



(a) e_c^* (b) e_d^*

(c) e_c^* (d) e_d^*

FIGURE 6.7: Pointwise aggregated equivalence of triangular fuzzy numbers with rational generators ($\nu = 0.6$)

Chapter 7

Main Results and Further Work

In this final chapter, I will conclude by outlining the progress made towards the goal described in the introduction. I will also suggest some future research directions that could provide the next steps along the path to a practical and widely applicable system.

In the first part of the thesis (sections 2 and 3), results on new constructions of continuous aggregation functions were presented.

In Section 2, a generation method of aggregation functions from two given ones was examined. The so-called threshold construction method is based on an adequate scaling on the second variable of the initial operators. This construction can be usuful in fuzzy applications where the inputs have different semantic contents. The new type of aggregation function turned out to be monotone and continuous, having a right-neutral and idempotent element. Three possible ways of symmetrizations were studied, two of them using min-max operators and the third using uninorms. After proving the lack of associativity in all cases, the bisymmetry and all the other associativity-like equations known from the literature were studied. Relevant own publication pertaining to this section: [19].

In Section 3, new construction methods of uninorms with fixed values along the borders were discussed, and sufficient and necessary conditions were presented. Relevant own publication pertaining to this section: [20].

In the second part of the thesis, logical systems, more specifically, nilpotent logical systems were deeply studied. The class of nilpotent t-norms and t-conorms has preferable properties which make them more usable in building up logical structures. Among these properties are the fulfillment of the law of contradiction and the excluded middle, or the coincidence of the residual and the S-implication. Due to the fact that all continuous Archimedean (i.e. representable) nilpotent t-norms are isomorphic to the Łukasiewicz t-norm, the previously studied nilpotent systems were all isomorphic to the well-known Łukasiewicz-logic.

In Section 4, it was shown that a consistent logical system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic. After giving a characterization and a wide range of examples for negation operators, connective systems were studied, in which the conjunction, the disjunction and the negation are generated by bounded and normalized functions. Three negations can be naturally associated with the normalized generator functions, n_c, n_d and n. Necessary and sufficient conditions of the classification property (the excluded middle and the law of contradiction), the De Morgan law and consistency have been given. The question whether the three negations can differ from one another in a consistent system was thoroughly examined. The positive answer means that a consistent system generated by nilpotent operators is not necessarily isomorphic to Łukasiewicz logic. A system can be built up in a significantly different way, using more than one generator functions. This new type of nilpotent logical systems is called a bounded system, which has the advantage of three naturally derived negations. The fixpoints of these natural negations can be used for determining thresholds for different modifying words. It was shown that we get a system isomorphic to Łukasiewicz logic if and only if the three negations coincide. Relevant own publications pertaining to this section: [27].

In Section 5, implication operators in bounded systems were deeply examined and a wide range of examples was also presented. The concept of a weak ordering property was defined. Two different implications, i_c and i_d were introduced, both of which fulfill all the basic features generally required for implications. Relevant own publication pertaining to this section: [28].

In Section 6, three different types of equivalence operators in bounded systems were studied. After taking a closer look at the implication-based equivalences, the properties of the so-called dual equivalences were studied. Using these two types of equivalence operators, a new concept of aggregated equivalences was introduced, which proved to possess nice properties like threshold transitivity, T-transitivity and associativity. For applications in image processing, the overall equivalence of two grey level images was defined, and an important semantic meaning of the aggregated equivalences was given. Relevant own publication pertaining to this section: [29].

The main disadvantage of the Łukasiewicz operator family is the lack of differentiability, which would be necessary for numerous practical applications. Although most fuzzy applications (e.g. embedded fuzzy control) use piecewise linear membership functions due to their easy handling, there are significant areas, where the parameters are learned by a gradient based optimization method. In this case, the lack of continuous derivatives makes the application impossible. For example, the membership functions have to be differentiable for every input in order to fine tune a fuzzy control system by a simple gradient based technique.

This problem could be easily solved by using the so-called squashing function (see Dombi and Gera, [32]), which provides a solution to the above mentioned problem by a continuously differentiable approximation of the cut function. This approximation could be the next step along the path to a practical and widely applicable system, with the advantage of three naturally derived negation operators.

Appendix A

Tables

Table A.1: Power functions as normalized generators

	f_n	f_c	f_d	Classification	De Morgan	Remarks
4.2	x^2	$\sqrt{1-x}$	$\sqrt{x}$	✓	–	
4.4	x	$(1-x)^\alpha$	x^α	✓	✓	$0 < \alpha \leq 1$
4.29	x	$(1-x)^\alpha$	x^α	–	✓	$\alpha > 1$
4.5	x	$(1-x)^{\frac{1}{\log_{0.5}(1-\nu_c)}}$	$x^{\log_{\nu_d} 0.5}$	iff $\nu_d \leq 0.5$	iff $\nu_c + \nu_d = 1$	4.4 and 4.29 in terms of the neutral value
4.6	x^α	$1-x^\alpha$	x^α	✓	✓	$\alpha > 0$
4.7	x^α	$(1-x^\alpha)^{\frac{\beta}{\alpha}}$	x^β	✓	✓	$\beta \leq \alpha;\quad \alpha, \beta > 0$
4.8	x	$1-x^\alpha$	$1-(1-x)^\alpha$	✓	✓	$\alpha \geq 1$, $f_c + f_d > 1$ iff $\alpha > 1$

Table A.2: Power functions as normalized generators – logical connectives

	f_n	f_c	f_d	$n(x)$	$c(x,y)$	$d(x,y)$
4.2	x^2	$\sqrt{1-x}$	$\sqrt{x}$	$\sqrt{1-x^2}$	$1-\left[\sqrt{(1-x)}+\sqrt{(1-y)}\right]^2$	$\left[\sqrt{x}+\sqrt{y}\right]^2$
4.4	x	$(1-x)^\alpha$	x^α	$1-x$	$1-[(1-x)^\alpha+(1-y)^\alpha]^{\frac{1}{\alpha}}$	$[x^\alpha+y^\alpha]^{\frac{1}{\alpha}}$
4.29	x	$(1-x)^\alpha$	x^α	$1-x$	$1-[(1-x)^\alpha+(1-y)^\alpha]^{\frac{1}{\alpha}}$	$[x^\alpha+y^\alpha]^{\frac{1}{\alpha}}$
4.6	x^α	$1-x^\alpha$	x^α	$\sqrt[\alpha]{1-x^\alpha}$	$(1-[2-x^\alpha-y^\alpha])^{\frac{1}{\alpha}}$	$[x^\alpha+y^\alpha]^{\frac{1}{\alpha}}$
4.7	x^α	$(1-x^\alpha)^{\frac{\beta}{\alpha}}$	x^β	$\sqrt[\alpha]{1-x^\alpha}$	$\left(1-\left[(1-x^\alpha)^{\frac{\beta}{\alpha}}+(1-y^\alpha)^{\frac{\beta}{\alpha}}\right]^{\frac{\alpha}{\beta}}\right)^{\frac{1}{\alpha}}$	$\left[x^\beta+y^\beta\right]^{\frac{1}{\beta}}$
4.8	x	$1-x^\alpha$	$1-(1-x)^\alpha$	$1-x$	$(1-[2-x^\alpha-y^\alpha])^{\frac{1}{\alpha}}$	$1-[(1-x)^\alpha+(1-y)^\alpha-1]^{\frac{1}{\alpha}}$

TABLE A.3: Exponential functions as normalized generators

f_n	f_c	f_d	De Morgan law	Consistency
$\frac{a^x-1}{a-1}$	$\frac{(a+1-a^x)^{\log_a b}-1}{b-1}$	$\frac{b^x-1}{b-1}$	✓	Consistent for e.g. $a=0.5, b=0.7$ or $a=0.7, b=0.85$

TABLE A.4: Rational functions as normalized generators

	f_n	f_c	f_d	Classification	De Morgan law
4.38 and 4.40	$\frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{\nu_c}{1-\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$	$\nu_d < \nu < \nu_c$	$\left(\frac{1-\nu}{\nu}\right)^2 = \frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}$ $\nu = \frac{1}{1+\sqrt{\frac{\nu_c}{1-\nu_c}\frac{1-\nu_d}{\nu_d}}}$
4.2	$\nu = 0.5$	$\nu_c = 0.7$	$\nu_d = 0.3$	✓	–
4.29	$\nu = 0.6$	$\nu_c = 0.2$	$\nu_d = 0.36$	–	✓
4.9	$\nu = 0.25$	$\nu_c = 0.5$	$\nu_d = 0.1$	✓	✓

TABLE A.5: Rational functions as normalized generators – 3 negations

	$f(x)$ (normalized generator)	$f^{-1}(x)$	$1-f(x)$	negation
negation	$\frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu}{\nu}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu}{\nu}\frac{x}{1-x}}$	$n(x) = \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2\frac{x}{1-x}}$
conjunction	$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{1-\nu_c}{\nu_c}\frac{x}{1-x}}$	$\frac{1}{1+\frac{\nu_c}{1-\nu_c}\frac{1-x}{x}}$	$n_c(x) = \frac{1}{1+\left(\frac{1-\nu_c}{\nu_c}\right)^2\frac{x}{1-x}}$
disjunction	$\frac{1}{1+\frac{\nu_d}{1-\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{1-x}{x}}$	$\frac{1}{1+\frac{1-\nu_d}{\nu_d}\frac{x}{1-x}}$	$n_d(x) = \frac{1}{1+\left(\frac{1-\nu_d}{\nu_d}\right)^2\frac{x}{1-x}}$

Table A.6: Mixed types of normalized generator functions

	f_n	f_c	f_d	De Morgan law	Consistency
Rational and power	$\frac{1}{1+\frac{\nu}{1-\nu}\frac{1-x}{x}}$	$\left(\frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2\frac{x}{1-x}}\right)^\alpha$	x^α	✓	Consistent for e.g. $\alpha = 1, \nu = 0.8$ or $\alpha = 2, \nu = 0.9$
Power and exponential	x^α	$\frac{a^{(1-x^\alpha)^{\frac{1}{\alpha}}}-1}{a-1}$	$\frac{a^x-1}{a-1}$	✓	$a > 0,\ a \neq 1,$ $\alpha > 0.$ Consistent for e.g. $\alpha = 1,\ a = 0.5$

Table A.7: Properties of implications in bounded systems

	formula	NP	EP	IP	SN	CP	WOP	OP
$\mathbf{i_c} = i_R$	$f_c^{-1}\left[f_c(y) - f_c(x)\right]$	✓	✓	✓	✓ $n_c(x)$	✓	✓	✓
$\mathbf{i_d} = i_{S_d}$	$f_d^{-1}\left[1 - f_d(x) + f_d(y)\right]$	✓	✓	✓	✓ $n_d(x)$	✓	✓	✓
i_{S_n}	$f_d^{-1}\left[f_c(x) + f_d(y)\right]$	✓	✓	✓	✓ $n(x)$	✓	✓ $\tau_{n,d}(x)$	–
i_{S_c}	$f_d^{-1}\left[f_d(y) + f_d(n_c(x))\right]$	✓	✓	✓	✓ $n_c(x)$	✓	✓ $\tau_{c,d}(x)$	–

TABLE A.8: The main properties of equivalence operators

	Implication-based e_c, e_d	Dual $\bar{e}_c, \bar{e}_d$	Aggregated e_c^*, e_d^*
Compatibility	✓	✓	✓
Symmetry	✓	✓	✓
Reflexivity	✓	–	–
$e(x, n(x)) = 0$	–	✓	–
$e(\nu, \nu) = \nu$	–	–	✓
Monotonicity	✓	–	✓
Threshold transitivity	–	–	✓
Invariance	✓	✓	✓
$e(1, x) = x$	✓	✓	✓
$e(0, x) = n(x)$	✓	✓	✓
Associativity	–	–	✓
T-transitivity	✓	✓	✓

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