Óbuda University

Doctoral (Ph.D.) Thesis

Functional Inequalities on Riemann-Finsler Manifolds

by

Ágnes Mester

Advisor:
Professor Alexandru Kristály, Ph.D.

Doctoral School of Applied Informatics and Applied Mathematics

Budapest, June 2022

Assessment Committee:

Professor Emeritus Péter Tibor Nagy, Ph.D., D.Sc., Óbuda University

Associate Professor and Head of Department Zoltán Muzsnay, Ph.D., University of Debrecen

Public Defense Committee:

Chairman: Professor Emeritus Aurél Galántai, Ph.D., D.Sc., Óbuda University

Secretary: Senior Lecturer Léka Zoltán, Ph.D., Óbuda University

Opponents:

Associate Professor and Head of Department Zoltán Muzsnay, Ph.D., University of Debrecen

Professor Árpád Baricz, Ph.D., D.Sc., Óbuda University

Other Members:

Professor István Faragó, Ph.D., D.Sc., Eötvös Loránd University

Associate Professor Márton Kiss, Ph.D., Budapest University of Technology and Economics

Professor János Karátson, Ph.D., D.Sc., Eötvös Loránd University

Date of Public Defense:

2022

Functional Inequalities on Riemann-Finsler Manifolds

by

Ágnes Mester

Abstract

The theory of functional inequalities has a fundamental role in analysis, with major implications in mathematical physics and nonlinear partial differential equations. One of the main questions within this theory is the study of these inequalities on non-Euclidean spaces, which has as its scope the characterisation of the relationship between the geometry of the studied curved structures and the properties of the corresponding functional inequalities.

The purpose of this work is to demonstrate how certain geometric properties of Riemannian/ Finsler manifolds affect various functional inequalities that hold on these spaces. As a result, the main portion of the thesis is concerned with several Sobolev-type inequalities with or without singular terms which are available on different Riemannian/ Finsler manifolds. The focus of this study is to understand the geometric factors of some nonlinear phenomena which occur on these curved structures. In order to comprehend better the particular geometric/ anisotropic framework of these manifolds, a number of Riemannian and Finsler model spaces are also investigated.

The first part of the thesis is devoted to the study of three Randers-type Finsler manifolds, which serve as model spaces for several examples and counterexamples throughout the dissertation. The next part concerns compact Sobolev embeddings à la Berestycki-Lions on noncompact Riemannian manifolds and Randers spaces. Thirdly, a number of Hardy-type inequalities are studied on Finsler manifolds and, in particular, on Finsler-Hadamard manifolds.

The primary application of these functional inequalities involves the study of elliptic partial differential equations. Accordingly, the last part of the thesis presents the application of the obtained Sobolev embeddings in order to establish a multiplicity result concerning an elliptic problem by using variational methods.

Acknowledgements

First of all, I would like to express my appreciation to my academic advisor, Professor Alexandru Kristály. I am grateful for his helpful guidance and insightful advice, as well as for fostering my mathematical curiosity and dedication. I would also like to thank him for the opportunity of being a member of the Project 127926 supported by the National Research, Development and Innovation Fund of Hungary.

I am grateful for the support and guidance I received from the late Professor Csaba Varga, who had guided me on my academic journey all the way since my bachelor studies. Therefore, I would like to dedicate this work to him, in the memory for his attentive care towards his many students.

Above all, I would like to thank my parents for their infinite love and selfless support. Szeretnék köszönetet mondani édesanyámnak és édesapámnak, akik gyerekkorom óta támogattak a matematika szépségeinek megismerésében. Hálás vagyok a rengeteg szeretetért, odaadásért, bíztatásért és segítségért, amit tőlük kaptam és kapok mind a mai napig.

I am thankful to my brother, Attila, for being a fellow Ph.D. student of mine throughout this journey, but most importantly, for always being there for me when I needed it the most.

Last but not least, I would like to thank my fiancé, Tamás, for his continuous patience, love and encouragement. I am extremely grateful for the great deal of support I received from him during the writing of this work. I would also like to thank him for helping me out in the preparation of the illustrations.

"*Perhaps it is good to have a beautiful mind, but even greater gift is to discover a beautiful heart.*"

A Beautiful Mind, 2001

Dedicated to the memory of Professor Csaba Varga.

Contents

Chapter 1

Introduction

Functional inequalities and Sobolev spaces play an outstanding role in functional and geometric analysis, mathematical physics, the theory of partial differential equations and the calculus of variations. Since several functional inequalities represent the manifestation of certain natural mathematical and physical phenomena, the study of these inequalities is a remarkable area of mathematics in itself.

A significant class of Sobolev-type inequalities is represented by those available on non-Euclidean structures. Although these curved spaces are natural extensions of the standard Euclidean space, the theory of Sobolev spaces and functional inequalities on these structures is far from being elementary, as the geometry of the ambient space can have substantial effects on the properties of Sobolev spaces and inequalities in question.

The primary objective of this thesis is to present the effects that certain geometric aspects of Riemannian/ Finsler manifolds can have on different functional inequalities available on these spaces. Accordingly, the major part of the dissertation is devoted to various Sobolev-type inequalities on Riemannian/ Finsler manifolds, with an emphasis on the interplay between the different sufficient/ necessary geometric conditions regarding the underlying curved spaces and the corresponding functional inequalities. In addition, we also explore several Riemannian/ Finsler model spaces, and demonstrate the particular geometric phenomena associated with these manifolds by a variety of examples and counterexamples. In the last part of the thesis we show the applicability of the theoretical results in the field of elliptic partial differential equations (in short, PDEs).

The present work is based on the following papers:

[1] C. Farkas, A. Kristály, and Á. Mester. "Compact Sobolev embeddings on non-compact manifolds via orbit expansions of isometry groups". *Calculus of Variations and PDE* 60.4 (2021), Article no: 128.

[2] Á. Mester and A. Kristály. "A bipolar Hardy inequality on Finsler manifolds". *2019 IEEE 13th International Symposium on Applied Computational Intelligence and Informatics (SACI)* (2019), pp. 308–313.

[3] Á. Mester and A. Kristály. “Three isometrically equivalent models of the Finsler-Poincaré disk”. *2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics (SACI)* (2021), pp. 403–408.

[4] Á. Mester, I. R. Peter, and C. Varga. “Sufficient Criteria for Obtaining Hardy Inequalities on Finsler Manifolds”. *Mediterranean Journal of Mathematics* 18 (2021), Article no: 76.

The thesis contains seven chapters. Chapter 2 provides an introduction to the theory of Riemann-Finsler geometry, outlining the fundamental analogies and differences between Riemannian and Finsler manifolds.

Chapter 3 is devoted to a specific class of Finsler manifolds called Randers spaces. In this context, we present the isometry between two well-known Randers models, namely the 2-dimensional Funk model and the Finsler-Poincaré disk, while also describing their connections to Riemannian geometry. Then, we introduce a new Randers model in the form of the Finsler-Poincaré upper half plane, and prove the isometrical equivalence of the three Finsler manifolds in question. Finally, we discuss some surprising geometric phenomena which result from the latter isometries. This section is based on Mester and Kristály [3].

Chapter 4 presents compact Sobolev embeddings à la Berestycki-Lions [18] on non-compact Riemannian manifolds and Randers spaces. First, we give a general introduction concerning Sobolev inequalities and continuous and compact embeddings in the Euclidean setting and on complete Riemannian manifolds. Then, given a noncompact complete Riemannian manifold (M,g) with certain curvature restrictions, we introduce a so-called expansion condition concerning a group of isometries G of (M,g) that characterizes the coerciveness of G in the sense of Skrzypczak and Tintarev [111]. Under this particular expansion condition, we prove compact Sobolev embeddings of the form $W^{1,p}_G(M) \hookrightarrow L^q(M)$ for the full range of admissible parameters (p,q), i.e., in the Sobolev, Moser-Trudinger and Morrey case, respectively. After this, we consider the case of noncompact Randers-type Finsler manifolds with finite reversibility constant, which turn out to inherit similar embedding properties as their Riemannian companions; the sharpness of such constructions is shown by means of the Funk model. This chapter is based on Farkas, Kristály, and Mester [1].

In Chapter 5 we establish various Hardy-type inequalities on forward complete Finsler manifolds. Adopting the arguments of D’Ambrosio and Dipierro [38] to the Finslerian context, we prove — among others — a Caccioppoli inequality, a Gagliardo-Nirenberg inequality and a Heisenberg-Pauli-Weyl uncertainty principle. Furthermore, we also obtain some Hardy inequalities on Finsler-Hadamard manifolds with finite reversibility constant. Finally, we study a Hardy inequality with multiple singularities on complete Finsler manifolds, obtaining the anisotropic counterpart of a Riemannian multipolar inequality due to Faraci, Farkas, and Kristály [48]. It turns out that the non-Riemannian properties of the ambient Finsler structure play a critical role in the validity of the studied inequalities, which is manifested by the dependence of the results on the so-called reversibility constant and uniformity constant of the Finsler manifold in question. This chapter is based on Mester, Peter, and Varga [4] and Mester and Kristály [2].

Chapter 6 is devoted to the application of the established functional inequalities in the theory of elliptic PDEs, by means of variational methods. More precisely, we show a multiplicity result concerning a quasilinear PDE involving the p-Finsler-Laplace operator, which is defined on a Randers space satisfying certain geometric assumptions. The proof is based on variational arguments, where the compact Sobolev embedding results established in Chapter 4 provide the means to verify essential properties of the energy functional associated with the studied problem, in order to apply certain minimization arguments. This section elaborates the proof of the multiplicity result given in Farkas, Kristály, and Mester [1].

Finally, Chapter 7 contains the formulation of the theses which summarize the most significant contributions of the present work.

Chapter 2

Preliminaries on Riemann-Finsler geometry

Riemannian geometry represents an extension of the classical Euclidean geometry, where the metric of the space is determined by a family of inner products, each defined on one of the tangent spaces of the underlying differentiable manifold. The mapping which associates to each point of the manifold a corresponding inner product in a differentiable manner is called a Riemannian metric. The pioneers of the development of Riemannian geometry were, among others, Lobachevsky, Bolyai, Gauss, Riemann and Beltrami.

Finsler geometry, in turn, arose from the need to generalize the Riemannian metric to asymmetric distances in case of anisotropic settings, see Finsler [53], Randers [102] and Matsumoto [87]. In this case, the metric of the ambient space is determined by the so-called Finsler structure, which defines Minkowski norms on every tangent space of the given manifold, all of which vary in a differentiable manner. Since the Minkowski norms are generally only positively homogeneous, the induced distance function on the space may not be symmetric. Therefore, Finsler geometry provides an appropriate framework to study anisotropic phenomena, having several applications in physics and other natural sciences, see Antonelli, Ingarden, and Matsumoto [9], Bao, Robles, and Shen [17], Caponio, Germinario, and Sánchez [28], Cheng and Shen [34], Gibbons and Warnick [58], Ishikawa [65], and references therein.

Note that the Finsler structure also induces a canonical inner product on the underlying differentiable manifold. However, unlike the Riemannian metric, this canonical metric depends not only on the points of the manifold, but also on the directions in the particular tangent spaces. Due to this considerable generalization, several reasonable objects and properties from Riemannian geometry convert to highly nonlinear phenomena in the case of general Finsler manifolds. On the other hand, many Riemannian notions do have their well-defined analogues in Finsler geometry, for example, affine connections, covariant derivatives, curvature, geodesics and distance function.

An expressive example of a Finsler-type geometry is given by Matsumoto's famous mountain slope metric, see Matsumoto [87]. In this model, on an inclined plane one measures the distance between two given points A and B by the time it takes to reach point B from point A, having constant velocity. Due to the presence of gravity, it turns out that the travelling time depends not just on the inclination of the plane, but also on the direction of movement. This directional dependence generates a non-Riemannian Finsler geometry, where the indicatrix corresponding to the Finsler norm is a limaçon, which depends on the angle of the plane and the velocity of the object. Evidently, the distance function associated to the Matsumoto metric will be asymmetric, meaning that $d_F(A, B) \neq d_F(B, A)$ unless $A = B$.

Another important Finslerian example is the metric defined by the Zermelo navigation problem, where one seeks time minimizing travel paths of an object having constant velocity on a given Riemannian manifold under the influence of an external force, such as wind, see Zermelo [131, 130] and Bao, Robles, and Shen [17]. In this case, the time minimizing geodesics are determined by a particular type of Finsler structure called Randers metric, and the induced Finsler manifold will be, in fact, a Randers space. The problem admits numerous generalizations and applications, see e.g., Caponio, Javaloyes, and Sánchez [29], Kopacz [71] and references therein.

Last but not least, we should highlight the well-known Finsler-Poincaré disk model, which simulates the time minimizing trajectories of an object on a circular region. Suppose that a force field directed towards the center of the disk acts on the body, see Bao, Chern, and Shen [15, Section 12.6]. In this peculiar model, it turns out that if the magnitude of the force is sufficiently large, the Finslerian distance from the boundary of the disk to the center will be finite, however, the distance from the center to the boundary will be infinite.

The purpose of this chapter is to give an overview of the fundamental notions of Finsler geometry, which are necessary for our further developments. Besides, we summarize the main analogies and differences between Riemannian manifolds and Finsler manifolds. For a comprehensive treatment of the subject, see Bao, Chern, and Shen [15], Ohta and Sturm [99] and Shen [109].

2.1 Riemannian vs. Finsler manifolds

Let M be a connected n-dimensional C^∞-differentiable manifold. The tangent bundle of M is the collection of all vectors tangent to M, i.e.,

$$TM = \bigcup_{x \in M} \{(x, v) : v \in T_xM\},$$

where T_xM denotes the tangent space of M at the point x.

Definition 2.1.1. The pair (M, F) is called a Finsler manifold, if $F : TM \to [0, \infty)$ is a continuous function such that

(i) $F \in C^\infty(TM \setminus \{0\})$;

(ii) $F(x;\lambda v) = \lambda F(x;v)$, for every $\lambda \geq 0$ and $(x;v) \in TM$;

(iii) the Hessian matrix

$$\left[g_{ij}(x;v)\right]_{i,j=\overline{1,n}} = \left[\frac{1}{2}\frac{\partial^2}{\partial v^i \partial v^j}F^2(x;v)\right]_{i,j=\overline{1,n}}$$

is positive definite for every $(x;v) \in TM \setminus \{0\}$, where $v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}$ in a local coordinate system $(x^i)_{i=\overline{1,n}}$.

The function F is called the Finsler structure on M. If, in addition, $F(x;\lambda v) = |\lambda| F(x;v)$ holds for all $\lambda \in \mathbb{R}$ and $(x;v) \in TM$, then the Finsler structure is symmetric and the Finsler manifold is called reversible. Otherwise, F is asymmetric and $(M;F)$ is said to be nonreversible.

Definition 2.1.2. The pair $(M;g)$ is called a Riemannian manifold, if g is a correspondence which associates to every point $x \in M$ an inner product $g_x : T_xM \times T_xM \to \mathbb{R}$ (i.e., a symmetric, bilinear, positive definite form), such that the functions $g_{ij}(x) := g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ are of class C^∞, for all $i,j = \overline{1,n}$. In this case g is called a Riemannian metric.

Note that if $(M;g)$ is a Riemannian manifold, then g induces a symmetric Finsler structure $F : TM \to [0,\infty)$ on M by

$$F(x;v) = \sqrt{g_x(v;v)}, \quad \forall (x;v) \in TM.$$

In this case, it turns out that the Hessian matrices $\left[g_{ij}(x;v)\right]_{i,j} = \left[g_{ij}(x)\right]_{i,j}$ do not depend on the tangent vector $v \in T_xM$, for every $x \in M$. Therefore, every Riemannian manifold is a reversible Finsler manifold.

In the following, unless otherwise stated, let $(M;F)$ denote an n-dimensional Finsler manifold.

2.2 Chern connection, geodesics and flag curvature

Let π^*TM be the pull-back tangent bundle of TM induced by the natural projection $\pi : TM\setminus\{0\} \to M$, i.e., π^*TM is the collection of all pairs $(y;w)$ with $y := (x;v) \in TM\setminus\{0\}$ and $w \in T_xM$. Then π^*TM admits a natural local basis defined by $\partial_i|_y := (y; \frac{\partial}{\partial x^i})$, and a canonical inner product induced by the Hessian matrices $[g_{ij}(x;v)]_{i,j=\overline{1,n}}$, i.e.,

$$g_y(\partial_i|_y, \partial_j|_y) = g_{(x;v)}(\partial_i|_y, \partial_j|_y) = g_{ij}(x;v), \quad \forall i,j = \overline{1,n}.$$

This naturally induced Riemannian metric g is called the fundamental tensor on π^*TM.

However, in contrast to the Levi-Civita connection in the Riemannian case, the above fundamental tensor does not induce a unique, metric compatible connection on the Finsler manifold (M,F). Nonetheless, among the connections on the pull-back tangent bundle π^*TM it is possible to choose a linear, torsion-free and almost metric-compatible connection called the Chern connection, see Bao, Chern, and Shen [15, Chapter 2]. The Chern connection induces the notion of covariant derivative and parallelism of a vector field along a curve. For example, let us denote by $D_v V$ the covariant derivative of a vector field V in the direction $v \in T_xM$. Then, a vector field $V = V(t)$ is parallel along a curve $\gamma = \gamma(t)$ if $D_{\dot\gamma} V = 0$.

A C^∞-differentiable curve $\gamma : [a,b] \to M$ is called a geodesic if its velocity field $\dot\gamma$ is parallel along the curve, i.e., $D_{\dot\gamma}\dot\gamma = 0$. The Finsler manifold is said to be forward (respectively, backward) complete if every geodesic segment $\gamma : [a,b] \to M$ can be extended to a geodesic defined on $[a,\infty)$ (respectively, on $(-\infty,b]$). In particular, (M,F) is called complete if it is forward and backward complete.

With the help of the Chern connection, one can also define the Chern curvature tensor R and the flag curvature K, see Bao, Chern, and Shen [15, Chapter 3]. For a fixed point $x \in M$, let $v,w \in T_xM$ be two linearly independent tangent vectors and $S = \mathrm{span}\{v,w\} \subset T_xM$. Then the flag curvature associated with the flag $(S;v)$ is defined as

$$\mathrm{K}(S;v) = \frac{g_y(\mathrm{R}(W,V)V,W)}{g_y(V,V)g_y(W,W) - g_y(V,W)^2},$$

where $y := (x,v) \in TM \setminus \{0\}$, $V := (y;v), W := (y;w) \in \pi^*TM$ and g is the fundamental tensor on π^*TM. Note that when (M,F) is a Riemannian manifold, the flag curvature reduces to the well-known sectional curvature which depends only on S.

We say that the flag curvature of (M,F) is bounded from above by some constant $c \in \mathbb{R}$ if $\mathrm{K}(S;v) \leq c$, for every $x \in M$ and every choice of $v,w \in T_xM$; this is denoted by $\mathrm{K} \leq c$. In particular, (M,F) is called a Finsler-Hadamard manifold if it is a simply connected, forward complete Finsler manifold having nonpositive flag curvature $\mathrm{K} \leq 0$. A Riemannian Finsler-Hadamard manifold is called a Cartan-Hadamard (or simply, Hadamard) manifold.

The Ricci curvature at the point $x \in M$ and in the direction $v \in T_xM$ is defined as

$$\mathrm{Ric}_x(v) = F^2(x,v)\sum_{i=1}^{n-1} \mathrm{K}(S_i;v),$$

where $\{e_1,\ldots,e_{n-1},\frac{1}{F(x,v)}v\}$ is an orthonormal basis of T_xM with respect to g, and $S_i = \mathrm{span}\{e_i,v\}$, for every $i = \overline{1,n-1}$. We say that the Ricci curvature of the manifold (M,F) is bounded from below by a constant $c \in \mathbb{R}$ if $\mathrm{Ric}_x(v) \geq c\, F^2(x,v)$ for all $(x,v) \in TM$; we denote this by $\mathrm{Ric}_M \geq c$.

2.3 Hausdorff volume form and distance function

The density function $\sigma_F : M \to [0, \infty)$ is defined by

$$\sigma_F(x) = \frac{\omega_n}{\mathrm{Vol}_e(B_x(1))},$$

where ω_n and $\mathrm{Vol}_e(B_x(1))$ denote the Euclidean volume of the n-dimensional Euclidean unit ball and the set

$$B_x(1) = \left\{(v^i) \in \mathbb{R}^n : F\left(x, \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\right) < 1\right\} \subset \mathbb{R}^n.$$

The canonical Hausdorff volume form $\mathrm{d}v_F$ on (M, F) is defined as

$$\mathrm{d}v_F(x) = \sigma_F(x) dx^1 \wedge \ldots \wedge dx^n,$$

see Shen [109, Section 2.2]. Note that in the thesis we may omit the parameter x for the sake of brevity.

The Finslerian volume of an open set $\Omega \subset M$ is given by $\mathrm{Vol}_F(\Omega) = \int_\Omega \mathrm{d}v_F(x)$. When $(M, F) = (M, g)$ is a Riemannian manifold, the Riemannian measure and Riemannian volume is denoted by $\mathrm{d}v_g$ and Vol_g, while in the particular n-dimensional Euclidean case, we simply use the notations $\mathrm{d}x$ and Vol_e.

The mean distortion of (M, F) is defined by

$$\tau : TM \setminus \{0\} \to (0, \infty), \qquad \tau(x, v) = \frac{\sqrt{\det\left[g_{ij}(x, v)\right]}}{\sigma_F(x)},$$

while the mean covariation is given by

$$\mathbf{S} : TM \setminus \{0\} \to \mathbb{R}, \quad \mathbf{S}(x, v) = \frac{d}{dt}\left(\ln \tau(\gamma(t), \dot\gamma(t))\right)\Big|_{t=0},$$

where γ is the geodesic with $\gamma(0) = x$ and $\dot\gamma(0) = v$. If $\mathbf{S}(x, v) = 0$ on all $TM \setminus \{0\}$, then we say that (M, F) has vanishing mean covariation and denote it by $\mathbf{S} = 0$.

The Finslerian distance function $d_F : M \times M \to [0, \infty)$ is defined by

$$d_F(x_1, x_2) = \inf \int_a^b F(\gamma(t), \dot\gamma(t))\, dt,$$

where $\gamma : [a, b] \to M$ is any piecewise differentiable curve such that $\gamma(a) = x_1$ and $\gamma(b) = x_2$. It is immediate that $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$ and that d_F verifies the triangle inequality. However, in general, the Finslerian distance function is not symmetric. In fact, we have that $d_F(x_1, x_2) = d_F(x_2, x_1)$, for every $x_1, x_2 \in M$, if and only if (M, F)

is a reversible Finsler manifold. In particular, the Riemannian distance function $d_g : M \times M \to [0, \infty)$ is symmetric.

Due to the asymmetry of the distance function d_F, one needs to define separately the forward and backward open geodesic balls of center $x_0 \in M$ and radius $\rho > 0$, namely,

$$B_F^+(x_0, \rho) = \{x \in M : d_F(x_0, x) < \rho\} \quad \text{and} \quad B_F^-(x_0, \rho) = \{x \in M : d_F(x, x_0) < \rho\}.$$

If $(M, F) = (M, g)$ is a Riemannian manifold, the forward and backward geodesic balls coincide and are simply given by the Riemannian geodesic ball

$$B_g(x_0, \rho) = \{x \in M : d_g(x_0, x) < \rho\}.$$

2.4 Polar transform and Legendre transform

The polar transform (or co-metric) $F^* : T^*M \to [0, \infty)$ is defined as the dual metric of F, namely

$$F^*(x, \xi) = \sup_{v \in T_xM \setminus \{0\}} \frac{\xi(v)}{F(x, v)},$$

where $T^*M = \bigcup_{x \in M} T_x^*M$ is the cotangent bundle of M and T_x^*M is the dual space of T_xM.

Since $F^{*2}(x, \cdot)$ is twice differentiable on $T_x^*M \setminus \{0\}$, one can define the Hessian (dual) matrix

$$\left[g_{ij}^*(x, \xi)\right]_{i,j=\overline{1,n}} = \left[\frac{1}{2}\frac{\partial^2}{\partial \xi_i \partial \xi_j} F^{*2}(x, \xi)\right]_{i,j=\overline{1,n}}$$

for every $\xi = \sum_{i=1}^{n} \xi_i dx^i \in T_x^*M \setminus \{0\}$ in a local coordinate system $(x^i)_{i=\overline{1,n}}$.

Using the strong convexity assumption on the Finsler structure F, the Legendre transform $J^* : T^*M \to TM$ is defined in the following way: for every $x \in M$ fixed, J^* associates to each $\xi \in T_x^*M$ the unique maximizer $v \in T_xM$ of the mapping

$$v \mapsto \xi(v) - \frac{1}{2}F^2(x, v).$$

Note that if $J^*(x, \xi) = (x, v)$, then

$$F(x, v) = F^*(x, \xi) \quad \text{and} \quad \xi(v) = F^*(x, \xi)F(x, v). \tag{2.1}$$

In local coordinates, for every $\xi = \sum_{i=1}^{n} \xi_i dx^i \in T_x^*M$, one has that

$$J^*(x, \xi) = \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\left(\frac{1}{2}F^{*2}(x, \xi)\right)\frac{\partial}{\partial x^i}.$$

2.5 Reversibility and uniformity constants

The reversibility constant of the Finsler manifold (M, F) is defined by the number

$$r_F = \sup_{x\in M} r_F(x) \in [1, \infty]; \quad \text{where} \quad r_F(x) = \sup_{v\in T_xM\setminus\{0\}} \frac{F(x, v)}{F(x, -v)};$$

and it measures how much the manifold deviates from being reversible, see Rademacher [101]. Note that $r_F = 1$ if and only if (M, F) is a reversible Finsler manifold. Also, if $r_F < \infty$, then the forward and backward completeness of (M, F) are equivalent, and in this case, we simply say that the Finsler manifold is complete, see Bao, Chern, and Shen [15, Section 6.6].

The uniformity constant of (M, F) is defined by the number

$$l_F = \inf_{x\in M} l_F(x) \in [0, 1]; \quad \text{where} \quad l_F(x) = \inf_{v,w,z\in T_xM\setminus\{0\}} \frac{g_{(x,w)}(v, v)}{g_{(x,z)}(v, v)};$$

which measures how much F deviates from being a Riemannian structure, see Egloff [45]. Indeed, $l_F = 1$ if and only if (M, F) is a Riemannian manifold, see Ohta [98].

By using the definition of l_F, it can be proved that

$$F^2(x, t\alpha + (1-t)\beta) \leq tF^2(x, \alpha) + (1-t)F^2(x, \beta) - l_F t(1-t)F^2(x, \beta - \alpha); \tag{2.2}$$

for every $x \in M$, $\alpha, \beta \in T_xM$ and $t \in [0, 1]$, see Ohta and Sturm [99]. Similarly, one can easily deduce that

$$l_F(x) r_F^2(x) \leq 1; \quad \forall x \in M.$$

Therefore, we have the following implication: if $l_F > 0$, then $r_F < \infty$.

2.6 Gradient, divergence and Finsler-Laplace operator

Let $u : M \to \mathbb{R}$ be a function of class C^1. The gradient of u is defined as $\nabla_F u : M \to TM$,

$$\nabla_F u(x) = J^*(x, Du(x));$$

where $Du(x) \in T_x^*M$ denotes the differential of u at the point x. Using the properties of the Legendre transform, it follows that

$$F^*(x, Du(x)) = F(x, \nabla_F u(x)); \quad \forall x \in M.$$

In local coordinates, one has that

$$Du(x)=\sum_{i=1}^{n}\frac{\partial u}{\partial x^i}(x)dx^i \quad \text{and} \quad \nabla_F u(x)=\sum_{i,j=1}^{n} g_{ij}(x,Du(x))\frac{\partial u}{\partial x^i}(x)\frac{\partial}{\partial x^j},$$

see Ohta and Sturm [99]. Note that the gradient operator ∇_F is usually nonlinear. However, when $(M,F)=(M,g)$ is a Riemannian manifold, ∇_F reduces to the Riemannian gradient operator ∇_g.

Here we recall the eikonal equation for the distance function d_F, see Shen [109, Lemma 3.2.3]. Namely, for every point $x_0\in M$, one has

$$F(x,\nabla_F d_F(x_0,x))=F^*(x,Dd_F(x_0,x))=Dd_F(x_0,x)(\nabla_F d_F(x_0,x))=1, \tag{2.3}$$

for all $x\in M\setminus(\{x_0\}\cup \mathrm{Cut}(x_0))$, where $\mathrm{Cut}(x_0)$ denotes the cut locus of the point x_0, see Bao, Chern, and Shen [15, Chapter 8].

Let $V:M\to TM$ be a differentiable vector field on M. The divergence of V is defined as $\mathrm{div}V:M\to\mathbb{R}$,

$$\mathrm{div}V(x)=\frac{1}{\sigma_F(x)}\sum_{i=1}^{n}\frac{\partial}{\partial x^i}(\sigma_F(x)V^i(x)),$$

where σ_F is the density function and $V=\sum_{i=1}^{n}V^i\frac{\partial}{\partial x^i}$ in a local coordinate system $(x^i)_{i=\overline{1,n}}$. In particular, we have the following divergence theorem:

$$\int_M u(x)\mathrm{div}V(x)dv_F(x)=-\int_M Du(x)\big(V(x)\big)dv_F(x), \tag{2.4}$$

see Ohta and Sturm [99].

For any $p\in\mathbb{N}, p\geq 2$, the p-Finsler-Laplace of a function $u\in C^2(M)$ is given by $\Delta_{F,p}u:M\to\mathbb{R}$,

$$\Delta_{F,p}u(x)=\mathrm{div}\big(F^*(x,Du(x))^{p-2}\nabla_F u(x)\big).$$

Note that in general, the operator $\Delta_{F,p}$ is nonlinear. When $p=2$, $\Delta_F:=\Delta_{F,2}$ is simply called the Finsler-Laplace operator. In particular, for a Riemannian manifold $(M,F)=(M,g)$, $\Delta_F=\Delta_g$ is the usual Laplace-Beltrami operator, see Hebey [63, p. 9].

The divergence theorem implies that

$$\int_M v(x)\Delta_{F,p}u(x)\,dv_F(x)=-\int_M F^*(x,Du(x))^{p-2}Dv(x)\big(\nabla_F u(x)\big)\,dv_F(x), \tag{2.5}$$

for all $u,v\in C_0^\infty(M)$.

2.7 Sobolev spaces on Finsler manifolds

Let $1 \leq p < \infty$ and $\Omega \subset M$ be an open set.

On the Finsler manifold (M,F), the spaces $L^p_{\mathrm{loc}}(\Omega)$ and $W^{1,p}_{\mathrm{loc}}(\Omega)$ can be defined in a natural manner, similarly to the Euclidean setting, see Brezis [22, p. 106]. In this case, these spaces turn out to be independent of the Finsler structure F. The Sobolev spaces on (M,F), however, are determined by the choice of F and the measure defined on M. While it is possible to use an arbitrary measure to define Sobolev spaces (see Ohta and Sturm [99]), in the thesis we shall use the canonical Hausdorff measure $\mathrm{d}v_F$.

The Sobolev spaces on Ω associated with (M,F) and $\mathrm{d}v_F$ are defined as

$$W^{1,p}_F(\Omega) = \left\{ u \in W^{1,p}_{\mathrm{loc}}(\Omega) : \int_\Omega F^{*p}(x, Du(x))\mathrm{d}v_F(x) < \infty \right\}.$$

It can be proved that $W^{1,p}_F(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to the (generally asymmetric) norm

$$\|u\|_{W^{1,p}_F(\Omega)} = \left(\int_\Omega F^{*p}(x, Du(x))\, \mathrm{d}v_F(x) + \int_\Omega |u(x)|^p \, \mathrm{d}v_F(x) \right)^{\frac{1}{p}}, \tag{2.6}$$

which is also equivalent with the norm

$$\left(\int_\Omega F^{*p}(x, Du(x))\, \mathrm{d}v_F(x) \right)^{\frac{1}{p}} + \left(\int_\Omega |u(x)|^p \, \mathrm{d}v_F(x) \right)^{\frac{1}{p}}. \tag{2.7}$$

Therefore, we may use the two norms interchangeably.

The space $W^{1,p}_{0,F}(\Omega) \subset W^{1,p}_F(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p}_F(\Omega)}$. In the case $p = 2$, we use the notations $H^1_F(\Omega) := W^{1,2}_F(\Omega)$ and $H^1_{0,F}(\Omega) := W^{1,2}_{0,F}(\Omega)$. Clearly, when $(M,F) = (M,g)$ is a Riemannian manifold, the Sobolev spaces $W^{1,p}_F(\Omega)$ and $W^{1,p}_{0,F}(\Omega)$ coincide with the Sobolev spaces $W^{1,p}_g(\Omega)$ and $W^{1,p}_{0,g}(\Omega)$ associated with the Riemannian metric g, see Hebey [63].

Since F is not necessarily symmetric, the Sobolev norms (2.6) (or (2.7)) are generally asymmetric norms as well, turning the Sobolev spaces $(W^{1,p}_F(\Omega), \|\cdot\|_{W^{1,p}_F(\Omega)})$ and $(W^{1,p}_{0,F}(\Omega), \|\cdot\|_{W^{1,p}_F(\Omega)})$ into asymmetric normed (or seminormed) spaces. Nevertheless, one can define the symmetric counterparts of the above Sobolev norms, by using the symmetrized Finsler structure associated with F, i.e., $F_s : TM \to [0, \infty)$,

$$F_s(x,v) = \left(\frac{F^2(x,v) + F^2(x,-v)}{2} \right)^{\frac{1}{2}}, \quad \forall (x,v) \in TM.$$

In this case, the symmetric norms $\|\cdot\|_{W^{1,p}_{F_s}(\Omega)}$ induced by F_s are defined as

$$\|u\|_{W^{1,p}_{F_s}(\Omega)} = \left(\int_\Omega F_s{}^p(x, Du(x))\,\mathrm{d}v_F(x) + \int_\Omega |u(x)|^p\,\mathrm{d}v_F(x)\right)^{\frac{1}{p}}.$$

Then, due to Farkas, Kristály, and Varga [50], we have the following result.

Theorem 2.7.1. (Farkas, Kristály, and Varga [50, Theorem 1.1]) *Let (M, F) be a complete n-dimensional Finsler manifold with $r_F < \infty$ and $\Omega \subseteq M$ an open domain. Then $(W^{1,2}_{0,F}(\Omega), \|\cdot\|_{W^{1,2}_{F_s}(\Omega)})$ is a reflexive Banach space. Moreover, the norms $\|\cdot\|_{W^{1,2}_{F_s}(\Omega)}$ and $\|\cdot\|_{W^{1,2}_{F}(\Omega)}$ are equivalent, in particular, one has that*

$$\left(\frac{1+r_F^2}{2}\right)^{-\frac{1}{2}} \|u\|_{W^{1,2}_{F}(\Omega)} \leq \|u\|_{W^{1,2}_{F_s}(\Omega)} \leq \left(\frac{1+r_F{}^{-2}}{2}\right)^{-\frac{1}{2}} \|u\|_{W^{1,2}_{F}(\Omega)}, \quad \forall u \in W^{1,2}_{0,F}(\Omega).$$

Therefore, if (M, F) is a complete Finsler manifold with finite reversibility constant, then, for any open domain $\Omega \subseteq M$, $(W^{1,2}_{0,F}(\Omega), \|\cdot\|_{W^{1,2}_{F}(\Omega)})$ is a reflexive biBanach space (i.e., complete asymmetric normed space, see Cobzaş [36]), since its associated normed space $(W^{1,2}_{0,F}(\Omega), \|\cdot\|_{W^{1,2}_{F_s}(\Omega)})$ satisfies the properties enumerated in Theorem 2.7.1. Moreover, the former result is sharp in the sense that there exist complete Finsler-Hadamard manifolds having $r_F = \infty$ such that the associated Sobolev spaces $W^{1,2}_{0,F}(M)$ not only lack completeness, but they do not even admit a vector space structure, see Farkas, Kristály, and Varga [50] and Kristály and Rudas [81]; obviously, in these cases the norms $\|\cdot\|_{W^{1,2}_{F_s}(M)}$ and $\|\cdot\|_{W^{1,2}_{F}(M)}$ are not equivalent.

Using relations (2.4) and (2.5), the divergence operator and the p-Finsler-Laplace operator can be defined in a distributional sense as well. For this let $V : \Omega \to TM$ be a vector field. By definition, we say that $V \in L^1_{\text{loc}}(\Omega)$ if the function $F(V) : \Omega \to [0, \infty), F(V)(x) = F(x, V(x))$ satisfies that $F(V) \in L^1_{\text{loc}}(\Omega)$. Then, for every vector field $V \in L^1_{\text{loc}}(\Omega)$, the divergence of V is defined as $\text{div}V : \Omega \to \mathbb{R}$, such that

$$\int_\Omega \varphi(x)\text{div}V(x)\mathrm{d}v_F(x) = -\int_\Omega D\varphi(x)\big(V(x)\big)\mathrm{d}v_F(x), \tag{2.8}$$

for every test function $\varphi \in C_0^\infty(\Omega)$, see Ohta and Sturm [99].

Similarly, for every function $u \in W^{1,p}_{\text{loc}}(\Omega)$, $2 \leq p < \infty$, the p-Finsler-Laplace operator is defined as

$$\int_M \varphi(x)\Delta_{F,p}u(x)\,\mathrm{d}v_F(x) = -\int_M F^*(x, Du(x))^{p-2}\, D\varphi(x)\big(\nabla_F u(x)\big)\,\mathrm{d}v_F(x), \tag{2.9}$$

for all $\varphi \in C_0^\infty(M)$.

2.8 Comparison theorems

We conclude this chapter with some comparison principles, which play a crucial role when studying the global geometry of the Finsler manifold (M,F).

In order to present these results, for any $c \in \mathbb{R}$ fixed, let $\mathbf{s}_c, \mathbf{ct}_c : (0,\infty) \to \mathbb{R}$ be the functions defined as

$$\mathbf{s}_c(t) = \begin{cases} \sin(\sqrt{c}t), & \text{if } \ c > 0, \\ t, & \text{if } \ c = 0, \\ \sinh(\sqrt{-c}t), & \text{if } \ c < 0, \end{cases} \quad \text{and} \quad \mathbf{ct}_c(t) = \begin{cases} \sqrt{c}\cot(\sqrt{c}t), & \text{if } \ c > 0, \\ \frac{1}{t}, & \text{if } \ c = 0, \\ \sqrt{-c}\coth(\sqrt{-c}t), & \text{if } \ c < 0. \end{cases}$$

The Laplacian comparison principle for the Finslerian distance function reads as follows.

Theorem 2.8.1. (Laplacian comparison theorem, Wu and Xin [124, Theorem 5.1]) *Let (M,F) be a complete n-dimensional Finsler manifold with $\mathbf{S} = 0$ and flag curvature bounded from above, i.e., $K \leq c$ for some $c \in \mathbb{R}$. If $r : M \to \mathbb{R}$, $r(x) = d_F(x_0,x)$ denotes the distance function from a fixed point $x_0 \in M$, then*

$$\Delta_F r(x) \geq (n-1)\mathbf{ct}_c(r(x)),$$

for every point $x \in M \setminus (\{x_0\} \cup \mathrm{Cut}(x_0))$.

Next, for any fixed numbers $c \in \mathbb{R}, n \in \mathbb{N}$ and $\rho > 0$, let

$$V_{c,n}(\rho) = n\omega_n \int_0^\rho \mathbf{s}_c^{n-1}(t)\mathrm{d}t$$

be the volume of the Riemannian geodesic ball of radius $\rho > 0$ in the n-dimensional space form having constant sectional curvature c (i.e., either the hyperbolic space $\mathbb{H}_c^n$ when $c < 0$, or the Euclidean space $\mathbb{R}^n$ when $c = 0$, or the n-dimensional sphere $\mathbb{S}_c^n$ when $c > 0$), see do Carmo [30, Chapter 8] and Chavel [33, Chapter III]. Then, for every $x \in M$, one has that

$$\lim_{\rho \searrow 0} \frac{\mathrm{Vol}_F(B_F^+(x,\rho))}{V_{c,n}(\rho)} = \lim_{\rho \searrow 0} \frac{\mathrm{Vol}_F(B_F(x,\rho))}{V_{c,n}(\rho)} = 1, \tag{2.10}$$

see Shen [110].

Finally, let us recall the following Bishop-Gromov-type volume comparison results on Finsler manifolds.

Theorem 2.8.2. (Volume comparison theorem I, Shen [110, Theorem 1.1]) *Let (M,F) be a complete n-dimensional Finsler manifold with $\mathbf{S} = 0$ and $\mathrm{Ric}_M \geq (n-1)c$ for some $c \in \mathbb{R}$. Then the function*

$$\rho \mapsto \frac{\mathrm{Vol}_F(B_F^+(x,\rho))}{V_{c,n}(\rho)}, \quad \rho > 0$$

is non-increasing, for every $x \in M$. In particular, by (2.10)*, one has that*

$$\mathrm{Vol}_F(B_F^+(x,\rho)) \leq V_{c,n}(\rho), \quad \textit{for all } \rho > 0 \textit{ and } x \in M. \tag{2.11}$$

Theorem 2.8.3. (Volume comparison theorem II, Wu and Xin [124, Theorem 6.1]) *Let (M,F) be a complete n-dimensional Finsler manifold with $\mathbf{S} = 0$ and $K \leq c$ for some $c \leq 0$. Then the function*

$$\rho \mapsto \frac{\mathrm{Vol}_F(B_F^+(x,\rho))}{V_{c,n}(\rho)}, \quad \rho > 0$$

is non-decreasing, for every $x \in M$. In particular, from (2.10) *it follows that*

$$\mathrm{Vol}_F(B_F^+(x,\rho)) \geq V_{c,n}(\rho), \quad \textit{for all } \rho > 0 \textit{ and } x \in M. \tag{2.12}$$

Chapter 3

Three isometrical models of Finsler manifolds

In this chapter we present three isometrically equivalent examples of Finsler manifolds, all of which belong to a particular class called Randers spaces. Roughly speaking, a Randers metric shows up as the perturbation of a Riemannian metric with a linear form. Therefore, Randers spaces represent a natural layer of generalization between Riemannian manifolds and general Finsler manifolds.

Due to the particular form of a Randers metric, the different geometric quantities of the ambient space turn out to be explicitly computable, making Randers spaces one of the most fundamental non-Riemannian family of Finsler manifolds. For mathematicians, they supply a collection of models for studying various non-Riemannian objects and quantities such as curvature, geodesics and connections, see Cheng and Shen [34] or Bao, Chern, and Shen [15, Chapter 11]. For those interested in the natural sciences, Randers spaces demonstrate a great potential of applicability in optics, thermodynamics, mathematical ecology and economics, see Antonelli, Ingarden, and Matsumoto [9], Dehkordi [39], Gibbons and Warnick [58] and references therein. In particular, they have received much attention lately among geometers due to their connection to the famous Zermelo navigation problem, see Zermelo [131, 130].

In what follows, first we review the definition of Randers spaces. Then we present three different Randers models and discuss their fundamental geometric properties, as well as their connection to Riemannian geometry. Finally, we prove that these spaces are actually all isometrically equivalent, and describe some interesting consequences of this property. This chapter summarizes the results of Mester and Kristály [3].

3.1 Randers spaces: definition and properties

Randers spaces were first studied by Randers [102], motivated by the necessity to model asymmetrical metrics and other anisotropic phenomena in physical applications. Randers metrics represent one of the simplest class of Finsler structures, as they are defined as an adequate perturbation of a Riemannian metric.

We say that the pair (M, F) is a Randers space if (M, g) is a Riemannian manifold and the Finsler structure $F : TM \to \mathbb{R}$ is defined by

$$F(x, v) = \sqrt{g_x(v, v)} + \beta_x(v), \quad \forall x \in M, \ v \in T_xM, \tag{3.1}$$

where β_x denotes a 1-form on M such that

$$|\beta_x|_g := \sqrt{g_x^*(\beta_x, \beta_x)} < 1, \quad \forall x \in M.$$

Here g^* denotes the co-metric of g, i.e., g_x^* is defined as the inverse of the symmetric, positive definite matrix g_x. In the above case F is called a Randers metric.

Evidently, every Riemannian manifold can be regarded as a Randers space with $\beta_x = 0, \forall x \in M$. Furthermore, it can be shown that every Randers space is a Finsler manifold, see Bao, Chern, and Shen [15, Section 1.3 C]. Also, the Finsler metric F from (3.1) is symmetric if and only if $\beta_x = 0, \forall x \in M$, which means that (M, F) coincides with the original Riemannian manifold (M, g). Finally, it can be proved that every Randers space can be obtained as the solution to the Zermelo navigation problem for a suitable choice of g and β_x, see Bao and Robles [16], Bao, Robles, and Shen [17], and Shen [108].

Due to the particular form of the Randers metric F from definition (3.1), several geometric notions of the corresponding manifold can be explicitly calculated. For example, the polar transform of F is given by

$$F^*(x, \xi) = \frac{\sqrt{g_x^{*2}(\xi, \beta_x) + (1 - |\beta_x|_g^2)|\xi|_g^2} - g_x^*(\xi, \beta_x)}{1 - |\beta_x|_g^2}, \quad \forall (x, \xi) \in T^*M. \tag{3.2}$$

The Hausdorff volume form dv_F on the Randers space (M, F) is

$$dv_F(x) = \left(1 - |\beta_x|_g^2\right)^{\frac{n+1}{2}} dv_g, \tag{3.3}$$

where dv_g denotes the canonical Riemannian volume form induced by the Riemannian metric g, and n is the dimension of M, see Cheng and Shen [34].

The reversibility constant of (M, F) is given by

$$r_F = \sup_{x \in M} r_F(x), \quad \text{where} \quad r_F(x) = \frac{1 + |\beta_x|_g}{1 - |\beta_x|_g}, \tag{3.4}$$

while the uniformity constant associated with F is

$$l_F = \inf_{x \in M} l_F(x), \quad \text{where} \quad l_F(x) = \left(\frac{1 - |\beta_x|_g}{1 + |\beta_x|_g}\right)^2, \tag{3.5}$$

see Yuan and Zhao [128].

Clearly, we have $r_F = 1$ (and $l_F = 1$, respectively) if and only if $\beta_x = 0, \forall x \in M$, i.e., $(M, F) = (M, g)$ is a Riemannian manifold.

3.2 Models of Randers spaces

In this section we present three explicit examples of Randers spaces, which turn out to be isometrically equivalent.

There exist two fundamental analytical models of Randers spaces in the literature:

(F): the Finslerian *Funk model* (see Cheng and Shen [34, Example 2.1.2] and Shen [109, Example 1.3.4]), which is defined as the perturbation of the well known Riemannian *Beltrami-Klein disk* (see Loustau [86, Section 6.2]);

(P): the *Finsler-Poincaré disk* (see Bao, Chern, and Shen [15, Section 12.6]) which is the Finslerian counterpart of the usual Riemannian *Poincaré disk model* (see Loustau [86, Section 8.1]).

Despite the popularity of these two Finsler models, the relationship among them is rarely discussed. This is even more peculiar if one considers the fact that these two Randers spaces are actually isometrically equivalent, meaning that there exists an isometric diffeomorphism between the two manifolds. To our knowledge, this equivalence is not well established in the literature. In fact, we found only one paper referring to the isometry map given in polar coordinates from the 2-dimensional Finsler-Poincaré disk onto the Funk model, in the context of Zermelo's navigation problem, see Bao and Robles [16, p. 240]. Therefore, a side objective of the present chapter is to describe in more detail the isometry between the models (F) and (P).

However, what is even more intriguing is that, as it turns out, there exists a third model which is isometric to the previous two. This phenomenon can be suspected from the properties of the hyperbolic model spaces. It is well known that the Riemannian counterparts of the models (F) and (P), i.e., the Beltrami-Klein disk and the Poincaré disk, are all isometric to the Poincaré upper half plane, see Cannon et al. [26]. Considering this fact, we introduce a new 2-dimensional analytic Finsler model, namely

(H): the *Finsler-Poincaré upper half plane*, which turns out to be precisely the Randers-type perturbation of the Riemannian *hyperbolic upper half plane model*, see Loustau [86, Section 8.2] or Stahl [114, Chapter 4].

In what follows, we present in detail the three Randers models in question, then we prove the isometrical equivalence of the three models. For simplicity of presentation, we restrict ourselves to dimension 2.

In the sequel we use the following notations:

- $\mathbb{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$ denotes the 2-dimensional Euclidean open unit disk;
- $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ represents the Euclidean upper half plane;
- $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the standard Euclidean norm and inner product on $\mathbb{R}^2$.

3.2.1 The Funk model

The Finslerian Funk model is given by the pair $(\mathbb{D}, F_F)$, where the Funk metric $F_F : \mathbb{D} \times \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$F_F(x, v) = \frac{\sqrt{(1 - |x|^2)|v|^2 + \langle x, v\rangle^2}}{1 - |x|^2} + \frac{\langle x, v\rangle}{1 - |x|^2}, \tag{3.6}$$

for all $(x, v) \in T\mathbb{D}$, see Cheng and Shen [34, Example 2.1.2] or Shen [109, Example 1.1.2 & 1.3.4].

Therefore, the Funk metric F_F can be viewed as the perturbation of the norm

$$(\alpha_F)_x(v) = \frac{\sqrt{(1 - |x|^2)|v|^2 + \langle x, v\rangle^2}}{1 - |x|^2} \tag{3.7}$$

with the 1-form

$$(\beta_F)_x(v) = \frac{\langle x, v\rangle}{1 - |x|^2}. \tag{3.8}$$

In fact, by omitting the 1-form $(\beta_F)_x$ in definition (3.6), we recover the Riemannian Klein metric on the unit disk $\mathbb{D}$. This metric induces the well-known Beltrami-Klein model, which is a hyperbolic model space of constant sectional curvature -1, where the geodesics are Euclidean straight lines in $\mathbb{D}$, see Loustau [86, Section 6.2].

It can be proved that $(\mathbb{D}, F_F)$ is a non-reversible Randers space having constant negative flag curvature $-\frac{1}{4}$, see Shen [109, Example 9.2.1]. Furthermore, it turns out that

$$|(\beta_F)_x|_{\alpha_F} = |x| < 1, \quad \forall x \in \mathbb{D}, \tag{3.9}$$

see Kristály and Rudas [81, Section 2.2], therefore, relation (3.4) implies that the reversibility constant of the manifold is $r_{F_F} = +\infty$. In particular, this results in the asymmetry of the induced distance function d_{F_F}, e.g., we have that

$$\lim_{x \to \partial\mathbb{D}} d_{F_F}(x, 0) = \ln 2 \quad \text{but} \quad \lim_{x \to \partial\mathbb{D}} d_{F_F}(0, x) = +\infty,$$

see Cheng and Shen [34, Example 2.1.2] or Shen [109, Example 1.1.2]. Note that the trajectories traced out by the geodesics of the Funk model coincide with the ones of the Beltrami-Klein disk, see Figure 3.1.

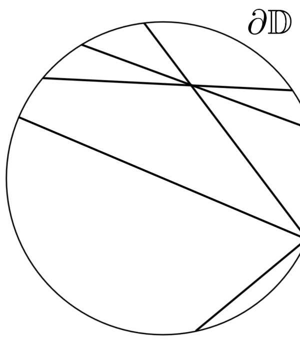


Figure 3.1: The pictorial representation of the Funk model $(\mathbb{D}, F_F)$: the geodesics are straight lines (chords) with ideal endpoints on the boundary $\partial\mathbb{D}$, see Shen [109, Example 9.2.1].

3.2.2 The Finsler-Poincaré disk

The Finsler-Poincaré metric on the open disk $\mathbb{D}$ is defined as $F_P : \mathbb{D} \times \mathbb{R}^2 \to \mathbb{R}$,

$$F_P(x, v) = \frac{2|v|}{1 - |x|^2} + \frac{4\langle x, v\rangle}{1 - |x|^4}, \tag{3.10}$$

for every $(x, v) \in T\mathbb{D}$.

The Finsler manifold $(\mathbb{D}, F_P)$ is called the Finsler-Poincaré disk, comprehensively investigated by Bao, Chern, and Shen [15, Section 1.3 E & 12.6]. Again, the metric F_P can be constructed as the perturbation of the norm

$$(\alpha_P)_x(v) = \frac{2|v|}{1 - |x|^2} \tag{3.11}$$

with the 1-form

$$(\beta_P)_x(v) = \frac{4\langle x, v\rangle}{1 - |x|^4}. \tag{3.12}$$

Direct calculations yield that

$$|(\beta_P)_x|_{\alpha_P} = \frac{2|x|}{1 + |x|^2} < 1, \quad \forall x \in \mathbb{D},$$

therefore $(\mathbb{D}, F_P)$ is a non-reversible Randers space with $r_{F_P} = +\infty$ (see formula (3.4)). Furthermore, one can prove that the Finsler-Poincaré disk is only forward complete, having constant negative flag curvature $-\frac{1}{4}$, see Bao, Chern, and Shen [15, Section 12.6]

Omitting $(\beta_P)_x$ in (3.10) results in the usual Riemannian Poincaré model, which is another fundamental hyperbolic manifold of constant sectional curvature -1, see Loustau [86, Section 8.1]. It turns out that the geodesics of $(\mathbb{D}, F_P)$ trajectory-wise coincide with their Riemannian counterparts: namely, they consist of Euclidean circular arcs which intersect the boundary $\partial\mathbb{D}$ at Euclidean right angles, and Euclidean straight lines that contain the origin, see Figure 3.2. However, the distance function d_{F_P} induced by the

Finsler-Poincaré metric is highly asymmetrical, in particular, we have that

$$\lim_{x \to \partial\mathbb{D}} d_{F_P}(x, 0) = \ln 2 \quad \text{but} \quad \lim_{x \to \partial\mathbb{D}} d_{F_P}(0, x) = +\infty,$$

see Bao, Chern, and Shen [15, Section 12.6].

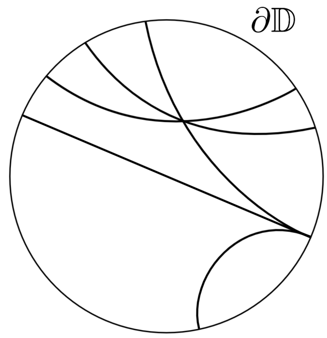


Figure 3.2: The pictorial representation of the Finsler-Poincaré disk model $(\mathbb{D}, F_P)$: the geodesics are straight lines going through the origin with ideal endpoints on the boundary $\partial\mathbb{D}$, or circular arcs perpendicular to the boundary, see Bao, Chern, and Shen [15, Section 12.6].

3.2.3 The Finsler-Poincaré upper half plane

Inspired by the previous Randers-type generalizations of the Beltrami-Klein disk and the Poincaré ball, we introduce the Finslerian counterpart of the Riemannian hyperbolic upper half plane (also referred to as the Poincaré half-plane model).

Let us define the Finsler metric $F_H : \mathbb{H} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$F_H(x, v) = \frac{|v|}{x_2} + \frac{\langle w(x), v \rangle}{x_2(4 + |x|^2)}, \tag{3.13}$$

for all $(x, v) \in T\mathbb{H}$, where $\mathbb{H}$ denotes the Euclidean upper half plane and

$$w(x) = (2x_1x_2, x_2^2 - x_1^2 - 4), \quad \forall x = (x_1, x_2) \in \mathbb{H}.$$

Then we call the pair $(\mathbb{H}, F_H)$ the Finsler-Poincaré upper half plane. In turns out that $(\mathbb{H}, F_H)$ is a Randers space, composed of the term

$$(\alpha_H)_x(v) = \frac{|v|}{x_2} \tag{3.14}$$

and the 1-form

$$(\beta_H)_x(v) = \frac{\langle w(x), v \rangle}{x_2(4 + |x|^2)}. \tag{3.15}$$

Indeed, α_H actually corresponds to the Lobachevsky metric of the Riemannian upper half plane, another standard hyperbolic model space with negative sectional curvature -1, see do Carmo [30, p. 73] or Loustau [86, Section 8.2].

Secondly, by direct calculation, one can show that

$$|(\beta_H)_x|_{\alpha_H} = \frac{|w(x)|}{4+|x|^2} < 1, \quad \forall x \in \mathbb{H}.$$

Moreover, we have that

$$\sup_{x\in\mathbb{H}} |(\beta_H)_x|_{\alpha_H} = \lim_{\substack{x\in\mathbb{H}\\ x_2\searrow 0}} \frac{|w(x)|}{4+|x|^2} = \lim_{\substack{x\in\mathbb{H}\\ x_2\searrow 0}} \frac{4+x_1^2}{4+|x|^2} = 1,$$

which yields that $r_{F_H} = +\infty$ (see relation (3.4)), i.e., $(\mathbb{H}, F_H)$ is non-reversible.

Regarding the geodesics of the newly constructed space, we can say the following: due to Bao, Chern, and Shen [15, p. 298], we know that if β_H is a closed 1-form, then the Finslerian geodesics are trajectory-wise identical to the geodesics of the underlying Riemannian manifold.

Since

$$\frac{\partial}{\partial x_2}\left(\frac{2x_1x_2}{x_2(4+|x|^2)}\right) = \frac{\partial}{\partial x_1}\left(\frac{x_2^2 - x_1^2 - 4}{x_2(4+|x|^2)}\right),$$

it follows that β_H is closed, therefore the trajectories of the geodesics of $(\mathbb{H}, F_H)$ and the Riemannian upper half plane coincide, see Figure 3.3.

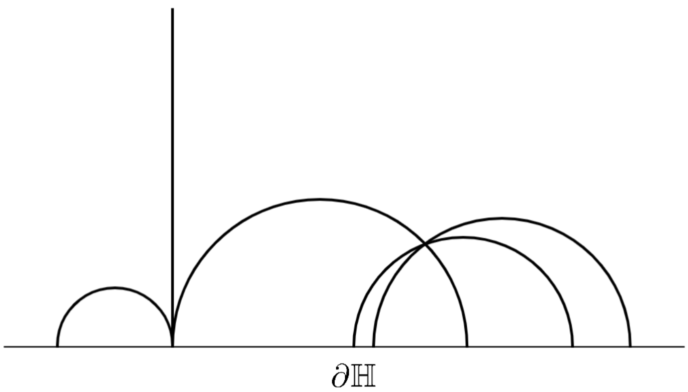


Figure 3.3: The pictorial representation of the Finsler-Poincaré upper half plane $(\mathbb{H}, F_H)$: the geodesics are straight Euclidean lines perpendicular to the x-axis and circular arcs intersecting the x-axis at Euclidean right angles (i.e., half-circles whose center is on the x-axis).

In conclusion, the Finsler manifold $(\mathbb{H}, F_H)$ is indeed the Randers-type perturbation of the Poincaré upper half plane, in the same fashion as the correspondences presented in Sections 3.2.1 & 3.2.2. Because of this analogous construction, one can prove that the

Finsler-Poincaré upper half plane defined by (3.13) is isometrically equivalent with the Funk model (3.6) and the Finsler-Poincaré disk (3.10). In order to put into context this result, we remark that Rutz and McCarthy [107] also considered a small perturbation of the Riemannian hyperbolic upper half plane, however, the metric obtained was not equivalent with the well-known models from Sections 3.2.1 or 3.2.2.

3.3 An isometry result

Theorem 3.3.1. *The Funk model* $(\mathbb{D}, F_F)$, *the Finsler-Poincaré disk* $(\mathbb{D}, F_P)$ *and the Finsler-Poincaré upper half plane* $(\mathbb{H}, F_H)$ *are isometrically equivalent.*

Before presenting the proof of Theorem 3.3.1, the following definition is in order:

Definition 3.3.1. Let (M_1, F_1) and (M_2, F_2) be two Finsler manifolds. If $f : M_1 \to M_2$ is a diffeomorphism such that

$$F_1(x, v) = F_2(f(x), Df_x(v)), \quad \forall (x, v) \in TM_1, \tag{3.16}$$

where Df_x denotes the differential of f at the point x, then f is an isometry between (M_1, F_1) and (M_2, F_2), and the manifolds (M_1, F_1) and (M_2, F_2) are said to be isometric.

Remark 3.3.1. Note that if f is an isometry between (M_1, F_1) and (M_2, F_2), then f is distance preserving in the following sense: if we consider the Finslerian distance functions d_{F_1} and d_{F_2} associated with the metrics F_1 and F_2 on M_1 and M_2, respectively, then the condition (3.16) implies that

$$d_{F_1}(x, y) = d_{F_2}(f(x), f(y)), \quad \forall x, y \in M_1.$$

Due to this property, we may call the manifolds (M_1, F_1) and (M_2, F_2) isometrically equivalent, meaning that there exists a distance preserving diffeomorphism between the respective spaces.

In the following subsections, we will prove Theorem 3.3.1 by giving the explicit form of the isometry functions between the Randers spaces in question.

3.3.1 Equivalence of $(\mathbb{D}, F_P)$ and $(\mathbb{D}, F_F)$

Proposition 3.3.1. *Let us consider the diffeomorphism*

$$f : \mathbb{D} \to \mathbb{D}, \quad f(x) = \frac{2x}{1 + |x|^2},$$

and its inverse

$$f^{-1} : \mathbb{D} \to \mathbb{D}, \quad f^{-1}(x) = \frac{x}{1 + \sqrt{1 - |x|^2}}.$$

Then f is an isometry between the Finsler-Poincaré disk $(\mathbb{D}; F_P)$ and the Funk model $(\mathbb{D}; F_F)$.

Proof. It is enough to show that

$$F_P(x; v) = F_F(f(x); Df_x(v)); \quad \forall (x; v) \in T\mathbb{D};$$

where Df_x denotes the differential of f at the point x.

Based on the definitions (3.6) – (3.8), we need to calculate the terms

$$(\alpha_F)_{f(x)}(Df_x(v)) = \frac{\sqrt{(1 - |f(x)|^2)|Df_x(v)|^2 + \langle f(x); Df_x(v)\rangle^2}}{1 - |f(x)|^2} \tag{3.17}$$

and

$$(\beta_F)_{f(x)}(Df_x(v)) = \frac{\langle f(x); Df_x(v)\rangle}{1 - |f(x)|^2}. \tag{3.18}$$

Given a point $x = (x_1; x_2) \in \mathbb{D}$, the differential function Df_x is determined by the Jacobian

$$Jf(x) = \frac{2}{(1 + |x|^2)^2}\begin{bmatrix} 1 + |x|^2 - 2x_1^2 & -2x_1x_2 \\ -2x_1x_2 & 1 + |x|^2 - 2x_2^2 \end{bmatrix}.$$

Then, for every $v \in T_x\mathbb{D} = \mathbb{R}^2$, we have

$$Df_x(v) = \frac{2}{(1 + |x|^2)^2}\begin{bmatrix} v_1(1 + |x|^2) - 2x_1\langle x; v\rangle \\ v_2(1 + |x|^2) - 2x_2\langle x; v\rangle \end{bmatrix}.$$

Expressing the terms

$$\begin{aligned}
1 - |f(x)|^2 &= \frac{(1 - |x|^2)^2}{(1 + |x|^2)^2}; \\
|Df_x(v)|^2 &= \frac{4}{(1 + |x|^2)^4}\left[(1 + |x|^2)^2|v|^2 - 4\langle x; v\rangle^2\right]; \\
\langle f(x); Df_x(v)\rangle &= 4\frac{1 - |x|^2}{(1 + |x|^2)^3}\langle x; v\rangle;
\end{aligned}$$

then substituting them into (3.17) and (3.18) yields

$$\begin{aligned}
(\alpha_F)_{f(x)}(Df_x(v)) &= \frac{(1 + |x|^2)^2}{(1 - |x|^2)^2} \\
&\quad \sqrt{\frac{4(1 - |x|^2)^2}{(1 + |x|^2)^6}[(1 + |x|^2)^2|v|^2 - 4\langle x; v\rangle^2] + 16\frac{(1 - |x|^2)^2}{(1 + |x|^2)^6}\langle x; v\rangle^2} \\
&= \frac{(1 + |x|^2)^2}{(1 - |x|^2)^2}\,\frac{2(1 - |x|^2)}{(1 + |x|^2)^2}|v| \\
&= \frac{2|v|}{1 - |x|^2} = (\alpha_P)_x(v)
\end{aligned}$$

and

$$\begin{aligned}(\;_F)_{f(x)}(Df_x(v)) &= \frac{(1+|x|^2)^2}{(1-|x|^2)^2}\,4\frac{1-|x|^2}{(1+|x|^2)^3}\langle x, v\rangle \\ &= \frac{4\langle x, v\rangle}{1-|x|^4} = (\;_P)_x(v),\end{aligned}$$

which concludes the proof. □

3.3.2 Equivalence of (D, F_F) and (H, F_H)

Proposition 3.3.2. *Let us consider the diffeomorphism*

$$g : \mathrm{D} \to \mathrm{H}, \quad g(x) = \left(\frac{2x_2}{1+x_1}, \frac{2\sqrt{1-|x|^2}}{1+x_1}\right)$$

with its inverse function

$$g^{-1} : \mathrm{H} \to \mathrm{D}, \quad g^{-1}(x) = \left(\frac{4-|x|^2}{4+|x|^2}, \frac{4x_1}{4+|x|^2}\right).$$

Then g is an isometry between the Funk model (D, F_F) and the Finsler-Poincaré upper half plane (H, F_H).

Proof. We prove that

$$F_F(x, v) = F_H(g(x), Dg_x(v)), \quad \forall (x, v) \in T\mathrm{D}.$$

By relations (3.14) and (3.15), for all $x \in \mathrm{D}$ and $v \in T_x\mathrm{D}$, one has

$$(\;_H)_{g(x)}(Dg_x(v)) = \frac{|Dg_x(v)|}{\frac{2\sqrt{1-|x|^2}}{1+x_1}} \tag{3.19}$$

and

$$(\;_H)_{g(x)}(Dg_x(v)) = \frac{\langle w(g(x)), Dg_x(v)\rangle}{\frac{2\sqrt{1-|x|^2}}{1+x_1}\,(4+|g(x)|^2)}, \tag{3.20}$$

where $w(y) = (2y_1y_2, y_2^2 - y_1^2 - 4)$, $\forall y = (y_1, y_2) \in \mathrm{H}$ (see definition (3.13)).

The Jacobian matrix of g is given by

$$\mathrm{J}g(x) = \frac{2}{(1+x_1)^2}\begin{bmatrix} -x_2 & (1+x_1) \\ -\frac{x_1 - x_2^2 + 1}{\sqrt{1-|x|^2}} & -\frac{x_2(1+x_1)}{\sqrt{1-|x|^2}} \end{bmatrix},$$

therefore, we have that

$$|Dg_x(v)| = \frac{2}{1+x_1} \cdot \sqrt{\frac{(1-|x|^2)|v|^2 + \langle x, v\rangle^2}{1-|x|^2}},$$

for every $x \in \mathbb{D}$ and $v \in T_x\mathbb{D} = \mathbb{R}^2$.

Using (3.19), it follows that

$$\begin{aligned}
(\alpha_H)_{g(x)}(Dg_x(v)) &= \frac{1+x_1}{2\sqrt{1-|x|^2}} \cdot \frac{2}{1+x_1} \cdot \sqrt{\frac{(1-|x|^2)|v|^2 + \langle x, v\rangle^2}{1-|x|^2}} \\
&= \frac{\sqrt{(1-|x|^2)|v|^2 + \langle x, v\rangle^2}}{1-|x|^2} = (\alpha_F)_x(v).
\end{aligned}$$

For the expression in (3.20), we use the following calculations:

$$\begin{aligned}
w(g(x)) &= \left(8\frac{x_2\sqrt{1-|x|^2}}{(1+x_1)^2},\ 8\frac{|x|^2+x_1}{(1+x_1)^2}\right), \\
4+|g(x)|^2 &= \frac{8}{1+x_1}, \quad \text{and} \\
\langle w(g(x)), Dg_x(v)\rangle &= \frac{16\langle x, v\rangle}{(1+x_1)^2\sqrt{1-|x|^2}}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
(\beta_H)_{g(x)}(Dg_x(v)) &= \frac{1+x_1}{2\sqrt{1-|x|^2}} \cdot \frac{1+x_1}{8} \cdot \frac{16\langle x, v\rangle}{(1+x_1)^2\sqrt{1-|x|^2}} \\
&= \frac{\langle x, v\rangle}{1-|x|^2} = (\beta_F)_x(v),
\end{aligned}$$

which completes the proof. □

3.3.3 Equivalence of $(\mathbb{H}, F_H)$ and $(\mathbb{D}, F_P)$

Proposition 3.3.3. *Let us consider the diffeomorphism*

$$h : \mathbb{H} \to \mathbb{D},\ h(x) = \left(\frac{4-|x|^2}{|x|^2+4x_2+4}, \frac{4x_1}{|x|^2+4x_2+4}\right),$$

and its inverse

$$h^{-1} : \mathbb{D} \to \mathbb{H}, h^{-1}(x) = \left(\frac{4x_2}{|x|^2+2x_1+1}, \frac{2-2|x|^2}{|x|^2+2x_1+1}\right).$$

Then h is an isometry between the Finslerian upper half plane $(\mathbb{H}, F_H)$ and the Finsler-Poincaré disk $(\mathbb{D}, F_P)$.

Proof. It is enough to prove that

$$F_H(x;v) = F_P(h(x); Dh_x(v)); \quad \forall (x;v) \in T\mathbb{H}.$$

From (3.11) and (3.12), we have

$$(\alpha_P)_{h(x)}(Dh_x(v)) = \frac{2|Dh_x(v)|}{1 - |h(x)|^2} \tag{3.21}$$

and

$$(\beta_P)_{h(x)}(Dh_x(v)) = \frac{4\langle h(x); Dh_x(v)\rangle}{1 - |h(x)|^4}. \tag{3.22}$$

The Jacobian of h can be written as

$$Jh(x) = \frac{4}{(|x|^2 + 4x_2 + 4)^2} \begin{bmatrix} 2x_1(x_2+2) & (x_2+2)^2 - x_1^2 \\ x_1^2 - (x_2+2)^2 & 2x_1(x_2+2) \end{bmatrix}.$$

First we do some preliminary computations: for every $x = (x_1; x_2) \in \mathbb{H}$ and $v = (v_1; v_2) \in T_x\mathbb{H} = \mathbb{R}^2$, one has

$$\begin{aligned} 1 - |h(x)|^2 &= \frac{8x_2}{|x|^2 + 4x_2 + 4}; \\ 1 + |h(x)|^2 &= 2\frac{|x|^2 + 4}{|x|^2 + 4x_2 + 4}; \\ 1 - |h(x)|^4 &= \frac{16x_2(|x|^2 + 4)}{(|x|^2 + 4x_2 + 4)^2}; \\ |Dh_x(v)| &= \frac{4|v|}{|x|^2 + 4x_2 + 4}; \end{aligned}$$

and

$$\begin{aligned} \langle h(x); Dh_x(v)\rangle = {}& \frac{4}{(|x|^2 + 4x_2 + 4)^3} \Big\{ (x_1^2 - (x_2+2)^2)\big(4x_1v_1 - (4 - |x|^2)v_2\big) \\ & + 2x_1(x_2+2)\big((4 - |x|^2)v_1 + 4x_1v_2\big) \Big\} \\ = {}& -4\,\frac{2x_1x_2v_1 + (x_2^2 - x_1^2 - 4)v_2}{(|x|^2 + 4x_2 + 4)^2}. \end{aligned}$$

Substituting these expressions into (3.21) and (3.22), it follows that

$$(\alpha_P)_{h(x)}(Dh_x(v)) = \frac{|v|}{x_2} = (\alpha_H)_x(v)$$

and

$$(\beta_P)_{h(x)}(Dh_x(v)) = 16\,\frac{2x_1x_2v_1 + (x_2^2 - x_1^2 - 4)v_2}{(|x|^2 + 4x_2 + 4)^2}\,\frac{(|x|^2 + 4x_2 + 4)^2}{16x_2(|x|^2 + 4)}$$
$$= \frac{2x_1x_2v_1 + (x_2^2 - x_1^2 - 4)v_2}{x_2(|x|^2 + 4)} = (\beta_H)_x(v),$$

which concludes the proof. □

Remark 3.3.2. (i) For the previous isometries we have that

$$h^{-1} = g \circ f,$$

i.e., the following diagram is commutative:

$$\begin{tikzcd} & (\mathbb{H}, F_H) \arrow[dl, shift left, "h"] \arrow[dr, shift left, "g^{-1}"] & \\ (\mathbb{D}, F_P) \arrow[ur, shift left, "h^{-1}"] \arrow[rr, shift left, "f"] & & (\mathbb{D}, F_F) \arrow[ul, shift left, "g"] \arrow[ll, shift left, "f^{-1}"] \end{tikzcd}$$

(ii) Theorem 3.3.1 is a natural extension of the isometrical equivalence of the Riemannian hyperbolic model spaces, more precisely, the Beltrami-Klein disk, the usual Poincaré disk and the Poincaré upper half plane, see Cannon et al. [26, p. 69]. In fact, the diffeomorphisms f, g and h coincide with the respective isometries between the Riemannian counterparts of the three models. This is also illustrated by the proofs of Propositions 3.3.1–3.3.3, where the norms $\alpha_F, \alpha_P, \alpha_H$ and the 1-forms $\beta_F, \beta_P, \beta_H$ turn out to be the pullbacks of one another by the corresponding isometries.

3.4 Consequences

An important byproduct of the isometry result given in Theorem 3.3.1 is the fact that all the metric related properties which hold on one particular model can be easily transferred to the other two manifolds by the appropriate isometry functions.

To give an interesting example, let us consider the first eigenvalue associated to the Finsler-Laplace operator Δ_F on the spaces $(\mathbb{D}, F_F)$, $(\mathbb{D}, F_P)$ and $(\mathbb{H}, F_H)$, respectively.

Given a general Finsler manifold (M, F), the first Dirichlet eigenvalue associated to Δ_F (also called the fundamental frequency) is defined as

$$\lambda_{1,F}(M) = \inf_{u \in H^1_{0,F}(M)\setminus\{0\}} \frac{\displaystyle\int_M F^{*2}(x, Du(x))\,\mathrm{d}v_F(x)}{\displaystyle\int_M u^2(x)\,\mathrm{d}v_F(x)},$$

where $H^1_{0,F}(M) = W^{1,2}_{0,F}(M)$ is the Sobolev space defined on the manifold (M, F), see Section 2.7.

According to Kristály [75, Theorem 1.3], in case of the Finslerian Funk model $(\mathbb{D}, F_F)$, we have that

$$\lambda_{1,F_F}(\mathbb{D}) = 0.$$

Combining this with the isometries given in Propositions 3.3.1 and 3.3.2, we can prove the gapless character of the first eigenvalue for the Randers spaces $(\mathbb{D}, F_P)$ and $(\mathbb{H}, F_H)$.

Corollary 3.4.1. *In case of the Finsler-Poincaré disk* $(\mathbb{D}, F_P)$ *and the Finsler-Poincaré upper half plane* $(\mathbb{H}, F_H)$*, we have*

$$\lambda_{1,F_P}(\mathbb{D}) = \lambda_{1,F_H}(\mathbb{H}) = 0.$$

These assertions are in sharp contrast with the result of McKean [90], which states that for every complete, n-dimensional, simply connected Riemannian manifold (M, g) having sectional curvature bounded above by $-\kappa^2$ $(\kappa > 0)$, one has the following spectral gap:

$$\lambda_{1,g}(M) \geq \frac{(n-1)^2}{4}\kappa^2.$$

In particular, in the case of the n-dimensional hyperbolic space $(\mathbb{H}^n, g_h)$ of constant curvature $-\kappa^2$ $(\kappa > 0)$, one has

$$\lambda_{1,g_h}(\mathbb{H}^n) = \frac{(n-1)^2}{4}\kappa^2,$$

see Chavel [33, p. 46]. Accordingly, the first Dirichlet eigenvalue of the Beltrami-Klein disk, the Riemannian Poincaré disk and the hyperbolic upper half plane is precisely $\frac{1}{4}$.

In conclusion, Corollary 3.4.1 provides new examples of simply connected, noncompact Finsler manifolds having constant negative flag curvature, such that their first Dirichlet eigenvalue vanishes. Considering the fact that the models in question represent some of the simplest non-Riemannian Finsler manifolds, these Randers spaces highlight the anisotropic phenomena that can occur in Finslerian settings.

Chapter 4

Sobolev-type inequalities without singular terms

Sobolev-type inequalities — or more generally, functional inequalities — play a crucial role in the theory of functional analysis, partial differential equations, mathematical physics, geometric analysis and calculus of variations. Indeed, the modern theory of nonlinear PDEs and boundary value problems (in short, BVPs) relies heavily on the theory of Sobolev spaces, since these are the natural function spaces in which one seeks the solution of such problems. Accordingly, there is a huge body of literature on Sobolev spaces and their applications; for a comprehensive presentation of the topic see the works of Adams and Fournier [5], Brezis [22], Evans [47], Maz'ya [89] and references therein.

Within this theory, a prominent class of Sobolev-type inequalities is provided by the ones defined on curved spaces. The systematic study of such relations originated in the 1970s with the works of Aubin [10, 12] and Cantor [27]. In fact, a particularly important incentive regarding this direction turned out to be the famous AB-program of Aubin [12], which had as its objective the determination of the best constants within such Sobolev inequalities on complete Riemannian manifolds. Since then, the study of functional inequalities on non-Euclidean structures has become a very active research area of geometric analysis.

It turns out that the properties of such inequalities deeply depend on the geometry of the ambient space, resulting in several surprising phenomena and challenging questions. Nevertheless, in the case of Riemannian manifolds, the theory of Sobolev spaces has undergone great development since the 1970s; for a comprehensive presentation of this topic, see Druet and Hebey [42], Hebey [63] and subsequent references. Moreover, the field has also established the grounds for new, thriving areas of research such as geometric analysis and optimal transport on general metric measure spaces, see Lott and Villani [85], Sturm [116, 117] and Villani [121].

Very recently, there has been a growing effort to extend the theory of Sobolev spaces and functional inequalities to Finsler manifolds, see e.g., Kristály [72], Kristály and Repovš [80], Ohta [96, 97], Ohta and Sturm [99] and Yuan, Zhao, and Shen [129]. However, due to the generally anisotropic nature of the Finsler metric, the adaptation of the standard Riemannian methods to the Finslerian setting requires critical analysis and careful attention, since several Riemannian objects and properties convert to highly nonlinear phenomena in the Finslerian framework, sometimes yielding unexpected results.

In this chapter, we first review some fundamental Sobolev inequalities available in the Euclidean space and on complete Riemannian manifolds. Then, we highlight some general, geometric conditions which are sufficient for obtaining compact Sobolev embeddings on not necessarily compact Riemannian manifolds. Next, we prove the validity of compact Berestycki-Lions-type embeddings for the full admissible range of Sobolev exponents on complete noncompact Riemannian manifolds which verify certain curvature restrictions and a so-called expansion condition. Finally, we consider the case of noncompact Randers spaces having finite reversibility constant, which turn out to inherit similar embedding properties as their Riemannian companions. We show the sharpness of the latter embedding results by a counterexample on the Finslerian Funk model. This chapter is based on Farkas, Kristály, and Mester [1].

4.1 Sobolev embeddings in the Euclidean case: a short overview

In the case of the Euclidean space $\mathbb{R}^n$, the classical Sobolev inequality was first obtained by Sobolev [113]. Later, a more straightforward proof was given by Gagliardo [55] and, independently, by Nirenberg [95].

Theorem 4.1.1. (Gagliardo-Nirenberg-Sobolev inequality) *Let $n \geq 2$ and $1 \leq p < n$. Then there exists a constant $C_{n,p} > 0$ depending only on n and p such that*

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}} \leq C_{n,p}\left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}}, \quad \forall u \in W^{1,p}(\mathbb{R}^n), \tag{4.1}$$

where $p^* := \frac{np}{n-p}$ denotes the critical Sobolev exponent of p.

In particular, inequality (4.1) directly implies the continuous Sobolev embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n),$$

for any $1 \leq p < n$, which in turn yields the validity of the embeddings $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$, for all $q \in [p, p^*]$, see, e.g., Brezis [22, Chapter 9.3].

Another spectrum of the Sobolev inequalities is given by the so-called Morrey inequality, which originates in the work of Morrey [92] and is directed towards the case $p > n$.

Theorem 4.1.2. (Morrey inequality) *Let $n \geq 2$ and $n < p < \infty$. Then there exists a constant $C_{n,p} > 0$ depending only on n and p such that for every $u \in W^{1,p}(\mathbb{R}^n)$, one has*

$$|u(x) - u(y)| \leq C_{n,p}|x - y|^{1-\frac{n}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p \mathrm{d}x \right)^{\frac{1}{p}}, \quad \text{a.e. } x, y \in \mathbb{R}^n. \tag{4.2}$$

In particular, the validity of inequality (4.2) results in the continuous Sobolev embedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$$

whenever $p > n$, see Brezis [22, Theorem 9.12].

By complementing the previous embedding results with the limiting case $p = n$ (see Brezis [22, Corollary 9.11]), one can obtain the full depiction of Sobolev embeddings in $\mathbb{R}^n$ (see Brezis [22, Corollary 9.14]).

Theorem 4.1.3. (Sobolev embedding theorem) *Let $\Omega \subseteq \mathbb{R}^n$ be an open set of class C^1 with bounded boundary ($n \geq 2$). If the parameters p, q satisfy one of the conditions*

(i) $1 \leq p < n$ *and* $p \leq q \leq p^*$;

(ii) $p = n$ *and* $p \leq q < \infty$;

(iii) $n < p < \infty$ *and* $q = \infty$,

then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ with continuous embedding.

Furthermore, in the case when Ω is bounded, most of the previous embeddings turn out to be compact, in the following sense (see Brezis [22, Theorem 9.16]).

Theorem 4.1.4. (Rellich-Kondrachov theorem) *Let $\Omega \subset \mathbb{R}^n$ be bounded and of class C^1 where $n \geq 2$. If the parameters p, q satisfy one of the conditions*

(i) $1 \leq p < n$ *and* $1 \leq q < p^*$;

(ii) $p = n$ *and* $p \leq q < \infty$;

(iii) $n < p < \infty$ *and* $q = \infty$,

then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

The compactness properties of the Sobolev embeddings turn out to be very important when applying them in the study of elliptic PDEs. Unfortunately, when Ω is unbounded – even if it has finite volume and smooth boundary – the aforementioned compact injections usually do not remain true, see, e.g., Adams and Fournier [5, Theorem 4.46]. For example, in the case when $\Omega = \mathbb{R}^n$, the dilation and translation of functions preclude such compactness results.

However, certain symmetry conditions may recover compactness. Indeed, in the case $1 \leq p \leq n$, it can be proved that the embedding $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is compact whenever $p < q < p^*$, where

$$W^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) = \left\{u \in W^{1,p}(\mathbb{R}^n) : u(\tau x) = u(x), \text{ for all } \tau \in O(n)\right\}$$

stands for the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^n)$, $O(n)$ denoting the orthogonal group of $\mathbb{R}^n$, see Berestycki and Lions [18], Lions [84], Cho and Ozawa [35], and Ebihara and Schonbek [44]. Furthermore, this compactness result also holds in the Morrey-Sobolev case, i.e., the embedding $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ is compact whenever $2 \leq n < p < \infty$, see Kristály [73]. In the following, these compactness results are referred to as Berestycki-Lions-type embeddings.

Geometrically, the argument behind the Berestycki-Lions compactness is based on a careful estimate of the functions at infinity. First, one can observe that the maximal number of mutually disjoint balls having a fixed radius and centered on the orbit $\{\tau x : \tau \in O(n)\}$ tends to infinity whenever $|x| \to \infty$. This expansiveness property of the balls combined with the fact that the Lebesgue measure is invariant with respect to translations implies that the radially symmetric functions rapidly decay to zero at infinity. This aspect is crucial to recovering compactness of Sobolev embeddings on unbounded domains, see Ebihara and Schonbek [44], Kristály [73] and Willem [123]. Moreover, this argument is in full concordance with the initial approach of Strauss [115].

4.2 Geometric conditions for compact embeddings

A natural extension of the Euclidean Sobolev embeddings is given by the Sobolev inequalities on Riemannian manifolds. In this setting, it turns out that the phenomena concerning Sobolev spaces are much more intricate, and the geometric properties of the ambient space play a critical role in the validity of certain Sobolev inequalities or embeddings. For a comprehensive treatment of the subject see the monograph of Hebey [63].

First, as someone may expect, compact Sobolev embeddings do remain valid on compact Riemannian manifolds. Concerning this, below we summarize the following results from Hebey [63, Theorems 2.6, 2.7 & 2.9].

Theorem 4.2.1. (Sobolev embeddings & Rellich-Kondrachov theorem on compact Riemannian manifolds) *Let (M,g) be an n-dimensional compact Riemannian manifold, $n \geq 2$. If the parameters p,q satisfy one of the conditions*

(i) $1 \leq p < n$ *and* $1 \leq q \leq p^*$;

(ii) $p = n$ *and* $1 \leq q < \infty$;

(iii) $n < p < \infty$ *and* $q = \infty$,

then the embedding $W_g^{1,p}(M) \hookrightarrow L^q(M)$ is continuous.

Moreover, the embedding $W_g^{1,p}(M) \hookrightarrow L^q(M)$ is compact, with the condition that $q \neq p^*$ in case (i).

On the other hand, the situation is quite different in the case of noncompact manifolds. For instance, it can be proved that for any integer $n \geq 2$, there exists an n-dimensional complete, noncompact Riemannian manifold (M, g) — possibly with finite volume — such that for any $p \in [1, n)$, $W_g^{1,p}(M) \not\subset L^{p^*}(M)$, which precludes any continuous or compact embedding, see Hebey [63, Propositions 3.4 & 3.5]. These counterexamples illustrate that in order to guarantee Sobolev embeddings in the noncompact setting, one needs to make additional assumptions regarding the geometry of the ambient space, see, e.g., Aubin [12] and Cantor [27].

It turns out that there are two main classes of complete, noncompact Riemannian manifolds which support continuous Sobolev embeddings: *Cartan-Hadamard manifolds* (or simply, Hadamard manifolds, which are simply connected, complete Riemannian manifolds with nonpositive sectional curvature), and Riemannian manifolds with *bounded geometry* (i.e., complete noncompact Riemannian manifolds with Ricci curvature bounded from below and positive injectivity radius).

In the case of Hadamard manifolds, one has the following Sobolev-type inequalities, see Hebey [63, Lemma 8.1 & Theorem 8.3].

Theorem 4.2.2. (Sobolev embeddings on Cartan-Hadamard manifolds) *Let (M, g) be an n-dimensional Cartan-Hadamard manifold with $n \geq 2$. If the parameters p, q satisfy one of the conditions*

(i) $1 \leq p < n$ *and* $q = p^*$;

(ii) $n < p < \infty$ *and* $q = \infty$,

then there exists a constant $C_{n,p} > 0$ depending only on n and p such that

$$\|u\|_{L^q(M)} \leq C_{n,p}\|\nabla_g u\|_{L^p(M)}, \quad \text{for all } u \in W_g^{1,p}(M). \tag{4.3}$$

In particular, the continuous embedding $W_g^{1,p}(M) \hookrightarrow L^q(M)$ holds.

In case (i), the sharp form of the Sobolev inequality (4.3) is also available whenever the so-called *Cartan-Hadamard conjecture* holds, see Muratori and Soave [93]. In this case, the best constant in (4.3) is exactly the optimal Euclidean constant of (4.1) obtained by Talenti [118]. The Cartan-Hadamard conjecture represents the isoperimetric inequality on Hadamard manifolds, with the optimal Euclidean isoperimetric constant; currently, the conjecture is proved in dimensions $n = 2, 3, 4$, see Weil [122], Kleiner [66] and Croke [37], and is an open question for $n \geq 5$. Regarding case (ii), sharp Morrey-Sobolev inequalities can also be proved on Cartan-Hadamard manifolds whenever the Cartan-Hadamard conjecture is true, see Kristály [77].

On Riemannian manifolds with bounded geometry the following embedding results hold, see Hebey [63, Proposition 3.6 & Theorem 3.4].

Theorem 4.2.3. (Sobolev embeddings on Riemannian manifolds with bounded geometry) *Let (M, g) be an n-dimensional Riemannian manifold with bounded geometry. If the parameters p, q satisfy one of the conditions*

(i) $1 \leq p < n$ *and* $q = p^*$*;*

(ii) $n < p < \infty$ *and* $q = \infty$*,*

then $W_g^{1,p}(M) \hookrightarrow L^q(M)$ *with continuous embedding.*

This bounded geometry condition can be slightly relaxed by assuming the existence of a lower bound for the volume of small balls, which is uniform with respect to their center, see Varopoulos [120]. Regarding this direction, Balogh and Kristály [14] obtained sharp Sobolev inequalities in case (i) on complete Riemannian manifolds having nonnegative Ricci curvature and so-called Euclidean volume growth, while Kristály, Mester, and Mezei [78] studied sharp Morrey-type inequalities in case (ii) in such geometric settings.

Concerning compact embeddings for radially symmetric functions, Hebey and Vaugon [64] established Berestycki-Lions-type results on complete Riemannian manifolds, assuming several geometric conditions regarding the ambient space and its isometry subgroup in question (see also Hebey [63, Theorems 9.5 & 9.6]). In order to sketch these results, let (M, g) be a complete n-dimensional Riemannian manifold, and let $\mathrm{Isom}_g(M)$ denote the isometry group of the manifold (M, g). Note that $\mathrm{Isom}_g(M)$ is a Lie group with respect to the compact open topology and it acts differentiably on M, see Myers and Steenrod [94]. Suppose that G is a compact subgroup of $\mathrm{Isom}_g(M)$. For any $x \in M$, let $O_G^x = \{\tau x : \tau \in G\}$ denote the G-*orbit* of the point x, where $\tau x := \tau(x)$ denotes the action of the element $\tau \in G$ on x. Then, under certain assumptions on the geometry of (M, g) and on the G-orbits of M, it can be proved that the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ is compact for a suitable range of the parameters p, q which depends on the orbits of G. Here, $W_G^{1,p}(M)$ denotes the set of G-invariant functions of $W_g^{1,p}(M)$, i.e.,

$$W_G^{1,p}(M) = \left\{u \in W_g^{1,p}(M) : u \circ \tau = u \text{ for all } \tau \in G\right\},$$

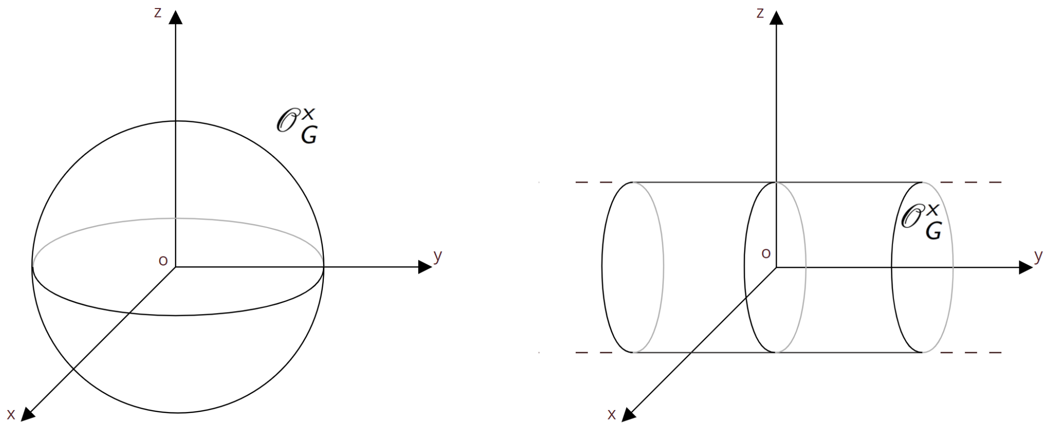
which turns out to be a closed subspace of $W_g^{1,p}(M)$, since G is a compact subgroup of $\mathrm{Isom}_g(M)$.

Skrzypczak and Tintarev [111] identified more general geometric conditions that are behind the compactness of the Sobolev embeddings of type $W_G^{1,p}(M) \hookrightarrow L^q(M)$ in the case when $1 < p < n$, n being the dimension of M (see also Tintarev [119]). Namely, they introduce the notion of coercive group action, and show that the coerciveness of the subgroup G provides a sufficient (and, under certain geometric assumptions on the underlying manifold, even necessary) condition for compactness.

A continuous action of a group G on a complete Riemannian manifold (M, g) is *coercive* (see Skrzypczak and Tintarev [111, Definition 1.2] or Tintarev [119, Definition 7.10.8]), if for every $t > 0$, the set

$$\mathcal{O}_t := \{x \in M : \operatorname{diam} \mathcal{O}_G^x \leq t\}$$

is bounded, see Figure 4.1



(A) The action of $SO(3)$ in $\mathbb{R}^3$ is coercive

(B) The planar rotation action of $SO(2)$ in $\mathbb{R}^3$ is not coercive

Figure 4.1: Examples of coercive and non-coercive group actions in the Euclidean space $(\mathbb{R}^3, e)$

On the one hand, if (M, g) is an n-dimensional Riemannian manifold with bounded geometry, and G is a compact connected subgroup of $\mathrm{Isom}_g(M)$, the coercive property of G turns out to be equivalent with the compactness of the embeddings $W_G^{1,p}(M) \hookrightarrow L^q(M)$ when $1 < p < n$ and $p < q < p^*$, see Tintarev [119, Theorem 7.10.12].

On the other hand, if (M, g) is an n-dimensional Cartan-Hadamard manifold, the situation is slightly more nuanced. Again, let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G) \neq \emptyset$, where

$$\mathrm{Fix}_M(G) = \{x \in M : \xi x = x \text{ for all } \xi \in G\}$$

denotes the *fixed point set* of the subgroup G on M. Skrzypczak and Tintarev prove that the coerciveness of G is equivalent with the fact that the set $\mathrm{Fix}_M(G)$ contains a single point of M, see Skrzypczak and Tintarev [111, Proposition 3.1]. Then, from one of these conditions they conclude the compactness of the embeddings $W_G^{1,p}(M) \hookrightarrow L^q(M)$ for every $1 < p < n$ and $p < q < p^*$.

In the light of these works, the purpose of the present chapter is twofold. First, we state the compact Sobolev embeddings of type $W_G^{1,p}(M) \hookrightarrow L^q(M)$ for the full admissible range of parameters, giving the Morrey counterpart (i.e., $n < p < \infty$) of the Sobolev case $1 < p < n$ described above. Secondly, we provide an alternative characterization of the coerciveness condition described by Skrzypczak and Tintarev, by studying the expansion of geodesic balls. This provides a more intuitive description of the geometric phenomena that allow Berestycki-Lions-type compactness; in addition, it connects the coercive property to the original approach of Strauss [115]. We distinguish two cases depending on the curvature of the ambient space, i.e., when (M,g) is a Hadamard manifold, or a Riemannian manifold with bounded geometry. Finally, we generalize the obtained compactness results to noncompact Randers spaces.

In order to present our results, let us introduce the following definitions. Given $n \in \mathbb{N}$ with $n \geq 2$, we say that $(p,q) \in (1,\infty) \times (1,\infty]$ is an *n-admissible pair* whenever one of the following conditions holds:

(S): $1 < p < n$ and $p < q < p^* = \frac{np}{n-p}$ (Sobolev case);

(MT): $p = n$ and $p < q < \infty$ (Moser-Trudinger case);

(M): $n < p < \infty$ and $q = \infty$ (Morrey case).

Let (M,g) be a complete n-dimensional Riemannian manifold, G a compact connected subgroup of $\mathrm{Isom}_g(M)$, and $x \in M$ a point on M. We denote by $m(x,\rho)$ the maximal number of mutually disjoint geodesic balls with radius ρ on the orbit O_G^x, i.e.,

$$m(x,\rho) = \sup\{k \in \mathbb{N} : \exists\, g_1,\dots,g_k \in G \text{ such that } B_g(g_i x,\rho) \cap B_g(g_j x,\rho) = \emptyset, \forall i \neq j\}, \tag{4.4}$$

where $B_g(y,\rho) = \{z \in M : d_g(y,z) < \rho\}$ is the usual geodesic ball in M and $d_g : M \times M \to [0,\infty)$ is the distance function induced by the Riemannian metric g.

For $\rho > 0$ and $x_0 \in M$ fixed, we introduce the following *expansion condition*:

$(\mathbf{EC})_G$ $m(x,\rho) \to \infty$ as $d_g(x_0,x) \to \infty$.

Clearly, condition $(\mathbf{EC})_G$ is independent of the choice of x_0.

By using the above expansion condition, we are able to give a characterization of the coercive action of the subgroup G.

4.3 Compact Sobolev embeddings on Hadamard manifolds

This section presents compact embeddings of isometry-invariant Sobolev functions on Hadamard manifolds. In order to do this, let $C(M)$ be the space of continuous functions $u : M \to [0,\infty)$ having compact support $D \subset M$, where D is smooth enough, u is of class C^2 in D and has only non-degenerate critical points in D. Based on classical Morse theory and density arguments, in this chapter we shall consider test functions $u \in C(M)$ in order to handle Sobolev inequalities.

Let $u \in C(M)$ and $\Omega \subset \mathrm{supp}(u) \subset M$ be an open set. Similarly to Druet, Hebey, and Vaugon [43], we may associate to the restriction of u to Ω, namely $u|_\Omega$, its *Euclidean rearrangement* function

$$u^*_\Omega : B_e(0,R_\Omega) \subset \mathbb{R}^n \to [0,\infty),$$

where $B_e(0;R^*)$ denotes the Euclidean open geodesic ball with center in the origin and radius R^*. The function u^* is radially symmetric, non-increasing in $|x|$, and for every $t \geq \inf_\Omega u$ is defined by

$$\mathrm{Vol}_e(\{x \in B_e(0;R^*) : u^*(x) > t\}) = \mathrm{Vol}_g(\{x \in \Omega : u(x) > t\}). \tag{4.5}$$

Here, Vol_e denotes the usual n-dimensional Euclidean volume and $R^* > 0$ is chosen such that $\mathrm{Vol}_g(\Omega) = \mathrm{Vol}_e(B_e(0;R^*)) = \omega_n R^{*n}$.

The following rearrangement properties are crucial in our further arguments.

Lemma 4.3.1. *Let (M,g) be an $n(\geq 2)$-dimensional Hadamard manifold. Let $u \in C(M)$ be a nonzero function, $\Omega \supseteq \mathrm{supp}(u)$, $\Omega \subseteq M$ be an open set, and $u^* : B_e(0;R^*) \to [0,\infty)$ its Euclidean rearrangement function. Then the following properties hold:*

(i) Volume-preservation: $\mathrm{Vol}_g(\mathrm{supp}(u)) = \mathrm{Vol}_e(\mathrm{supp}(u^*))$;

(ii) Cavalieri principle: *for every $q \in (0,\infty]$, $\|u\|_{L^q(\Omega)} = \|u^*\|_{L^q(B_e(0;R^*))}$;*

(iii) Pólya-Szegő inequality: *for every $p > 1$, we have*

$$\|\nabla_g u\|_{L^p(\Omega)} \geq \frac{C(n)}{n\omega_n^{\frac{1}{n}}}\|\nabla u^*\|_{L^p(B_e(0;R^*))}, \tag{4.6}$$

where $C(n) > 0$ is the Croke-constant (see Croke [37]), i.e., $C(2) = 1$ and

$$C(n) = (n\omega_n)^{1-\frac{1}{n}}\left((n-1)\omega_{n-1}\int_0^{\frac{\pi}{2}} \cos^{\frac{n}{n-2}}(t)\sin^{n-2}(t)\,\mathrm{d}t\right)^{\frac{2}{n}-1}, \quad \text{when } n \geq 3.$$

Note that $C(n) \leq n\omega_n^{\frac{1}{n}}$ for every $n \geq 3$, while equality holds if and only if $n = 4$.

The proof of the above lemma relies on suitable application of the co-area formula combined with the weak form of the isoperimetric inequality on Hadamard manifolds (for a similar proof, see Druet, Hebey, and Vaugon [43], and Kristály [77]).

Next, we prove the following Rellich-Kondrachov-type embedding, which is an expected result based on Aubin [11, Chapter 2]:

Lemma 4.3.2. *Let (M,g) be a complete n-dimensional Riemannian manifold. If $R > 0$, then the embedding $W_g^{1,p}(B_g(y;R)) \hookrightarrow L^q(B_g(y;R))$ is compact for every $y \in M$ and every n-admissible pair (p,q).*

Proof. Since $\overline{B_g(y;R)} \subset M$ is compact (due to the Hopf-Rinow theorem), the Ricci curvature is bounded from below (see Bishop and Crittenden [19, p. 166]) and the injectivity radius is positive on $\overline{B_g(y;R)}$ (see Klingenberg [67, Proposition 2.1.10] or Bao, Chern, and Shen [15, Chapter 8]).

Thus, we are in the position to use Hebey [63, Theorem 1.2]. Accordingly, for every $\varepsilon > 0$ there exists a harmonic radius $r_H > 0$, such that for every $z \in \overline{B_g(y,R)}$ one can find a harmonic coordinate chart $\varphi_z : B_g(z,r_H) \to \mathbb{R}^n$ such that $\varphi_z(z) = 0$ and the components (g_{jl}) of g in this chart satisfy

$$\frac{1}{1+\varepsilon}\delta_{jl} \le g_{jl} \le (1+\varepsilon)\delta_{jl}$$

as bilinear forms. Therefore, it follows that

$$\frac{1}{\sqrt{1+\varepsilon}} d_g(z,x) \le |\varphi_z(x)| \le \sqrt{1+\varepsilon}\, d_g(z,x), \text{ for all } x \in B_g(z,r_H). \tag{4.7}$$

Now let $0 < \delta < r_H$. Since $\overline{B_g(y,R)}$ is compact, there exists $L \in \mathbb{N}$ and $z_1,\dots,z_L \in \overline{B_g(y,R)}$ such that $\overline{B_g(y,R)} \subset \bigcup_{j=1}^{L} B_g(z_j,\delta)$. For every $z_j \in \overline{B_g(y,R)}$, $j = \overline{1,L}$, denote by

$$U_{z_j} := B_g(z_j,\delta) \cap B_g(y,R) \quad \text{and} \quad \Omega_{z_j} := \varphi_{z_j}\left(U_{z_j}\right) \subset \mathbb{R}^n,$$

thus $\left\{U_{z_j}\right\}_{j=\overline{1,L}}$ is a finite covering of $B_g(y,R)$.

First observe that for any $j \in \{1,\dots,L\}$ and $u \in W_g^{1,p}(B_g(y,R))$, on account of (4.7), we have that

$$\int_{U_{z_j}} |\nabla_g u|^p + |u|^p \mathrm{d}v_g \le \left(\frac{1}{\sqrt{1+\varepsilon}}\right)^{n+p}\left(\int_{\Omega_{z_j}} |\nabla(u \circ \varphi_{z_j}^{-1})|^p + |u \circ \varphi_{z_j}^{-1}|^p \mathrm{d}x\right). \tag{4.8}$$

We first focus on the (S) admissible case. Observe that

$$\int_{U_{z_j}} |u|^q \mathrm{d}v_g \le (1+\varepsilon)^{\frac{n}{2}} \int_{\Omega_{z_j}} |u \circ \varphi_{z_j}^{-1}|^q \mathrm{d}x. \tag{4.9}$$

Now, by the Euclidean continuous Sobolev embedding (see Theorem 4.1.3), for every $j \in \{1,\dots,L\}$ there exists a constant $C_{S,j}$ such that

$$\left(\int_{\Omega_{z_j}} |u \circ \varphi_{z_j}^{-1}|^q \mathrm{d}x\right)^{\frac{1}{q}} \le C_{S,j}\left(\int_{\Omega_{z_j}} |\nabla(u \circ \varphi_{z_j}^{-1})|^p + |u \circ \varphi_{z_j}^{-1}|^p \mathrm{d}x\right)^{\frac{1}{p}}. \tag{4.10}$$

Therefore, by (4.8), (4.9) and (4.10) we have that

$$\begin{aligned}
\|u\|_{L^q(B_g(y,R))} &\leq \sum_{j=1}^{L} \|u\|_{L^q(U_{z_j})} \leq (1+\varepsilon)^{\frac{n}{2q}} \sum_{j=1}^{L} \|u \circ \varphi_{z_j}^{-1}\|_{L^q(\Omega_{z_j})} \\
&\leq (1+\varepsilon)^{\frac{n}{2q}} \sum_{j=1}^{L} C_{S,j} \|u \circ \varphi_{z_j}^{-1}\|_{W^{1,p}(\Omega_{z_j})} \\
&\leq (1+\varepsilon)^{\frac{np+nq+pq}{2pq}} \sum_{j=1}^{L} C_{S,j} \|u\|_{W_g^{1,p}(U_{z_j})} \\
&\leq (1+\varepsilon)^{\frac{np+nq+pq}{2pq}} \sum_{j=1}^{L} C_{S,j} \, \|u\|_{W_g^{1,p}(B_g(y,R))},
\end{aligned}$$

which proves the validity of the continuous Sobolev embedding

$$W_g^{1,p}(B_g(y,R)) \hookrightarrow L^q(B_g(y,R))$$

in the case (S). Now we prove that the previous embedding is compact. To do this, let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W_g^{1,p}(B_g(y,R))$, and denote $u_k^j = u_k|_{U_{z_j}}$ for every $j \in \{1,\dots,L\}$. Using (4.8), we have that for every $j = \overline{1,L}$, the sequence $\tilde{u}_k^j = u_k \circ \varphi_{z_j}^{-1}$ is bounded in $W^{1,p}(\Omega_{z_j})$. By the Rellich-Kondrachov theorem one gets that there exists a subsequence of $(\tilde{u}_k^j)_k$ which is a Cauchy sequence in $L^q(\Omega_{z_j})$. Let $(u_m)_m$ be a subsequence of $(u_k)_k$ such that for any $j = \overline{1,L}$, $(\tilde{u}_m^j)_m$ is a Cauchy sequence in $L^q(\Omega_{z_j})$. Thus, applying (4.9), for any m_1, m_2 we obtain that

$$\|u_{m_1} - u_{m_2}\|_{L^q(B_g(y,R))} \leq \sum_{j=1}^{L} \|u_{m_1}^j - u_{m_2}^j\|_{L^q(U_{z_j})} \leq (1+\varepsilon)^{\frac{n}{2q}} \sum_{j=1}^{L} \|\tilde{u}_{m_1}^j - \tilde{u}_{m_2}^j\|_{L^q(\Omega_{z_j})},$$

hence $(u_m)_m$ is a Cauchy sequence in $L^q(B_g(y,R))$, which proves the claim.

The (MT) admissible case can be proved analogously, replacing (4.10) with the Euclidean Sobolev embedding when $p = n$ (see Theorem 4.1.3).

Finally, in the (M) case, we have that

$$\sup_{x\in\overline{B_g(y,R)}} |u(x)| = \max_{j=\overline{1,L}} \|u\|_{C^0(\overline{U_{z_j}})} = \max_{j=\overline{1,L}} \|u \circ \varphi_{z_j}^{-1}\|_{C^0(\overline{\Omega_{z_j}})}. \tag{4.11}$$

Again, by Theorem 4.1.3, for each $j \in \{1,\dots,L\}$ there exists a constant $C_{0,j}$ such that

$$\|u \circ \varphi_{z_j}^{-1}\|_{C^0(\overline{\Omega_{z_j}})} \leq C_{0,j} \, \|u \circ \varphi_{z_j}^{-1}\|_{W^{1,p}(\Omega_{z_j})},$$

thus this inequality together with (4.8) and (4.11) yields that

$$\begin{aligned}\sup_{x\in\overline{B_g(y,R)}}|u(x)| &\leq \max_{j=\overline{1,L}} C_{0,j}\|u\circ\varphi_{z_j}^{-1}\|_{W^{1,p}(\Omega_{z_j})}\\ &\leq \max_{j=\overline{1,L}} C_{0,j}(1+\varepsilon)^{\frac{n+p}{2p}}\|u\|_{W_g^{1,p}(U_{z_j})}\\ &\leq \left(\max_{j=\overline{1,L}} C_{0,j}\right)(1+\varepsilon)^{\frac{n+p}{2p}}\|u\|_{W_g^{1,p}(B_g(y,R))},\end{aligned} \tag{4.12}$$

which proves again that the continuous embedding holds. Now we prove that this injection is compact. To do this, consider a bounded set $A \subset W_g^{1,p}(B_g(y,R))$, i.e., there exists $M > 0$ such that $\|u\|^p_{W_g^{1,p}(B_g(y,R))} \leq M$ for all $u \in A$. Hence, by using (4.12), it follows that there exists $C_2 > 0$ such that $\|u\|_{C^0(\overline{B_g(y,R)})} \leq MC_2$ for all $u \in A$. Thus by the Arzelà-Ascoli theorem (see Aubin [12, Theorem 3.15]), we get that A is precompact in $C^0(\overline{B_g(y,R)})$, which concludes the proof. □

Now we are in the position to prove the following compact embedding result on Cartan-Hadamard manifolds:

Theorem 4.3.3. *Let (M,g) be an n-dimensional Hadamard manifold, and let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G) \neq \emptyset$. Then the following statements are equivalent:*

(i) G is coercive;

(ii) $\mathrm{Fix}_M(G)$ is a singleton;

(iii) $(\mathbf{EC})_G$ holds.

Moreover, from any of the above statements it follows that the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ is compact for every n-admissible pair (p,q).

Note that the equivalence between (i) and (ii) in Theorem 4.3.3 is proved by Skrzypczak and Tintarev [111, Proposition 3.1], from which they conclude the compactness of the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ for the admissible case (S); for a similar result in the case (MT), see Kristály [76]. Accordingly, the purpose of Theorem 4.3.3 is to characterize the coerciveness of G by the expansion condition $(\mathbf{EC})_G$, as well as to complement the admissible range of parameters with the Morrey case (M).

Proof of Theorem 4.3.3. $\underline{(i) \Leftrightarrow (ii)}$ This equivalence can be found in Skrzypczak and Tintarev [111, Proposition 3.1].

$\underline{(ii) \Rightarrow (iii)}$ Let $\mathrm{Fix}_M(G) = \{x_0\}$, $x_0 \in M$. Without loss of any generality, it is enough to prove that $m(\gamma(t),\rho) \to \infty$ as $t \to \infty$ for every unit speed geodesic $\gamma : [0,\infty) \to M$ emanating from $x_0 = \gamma(0)$; i.e., $\gamma(t) = \exp_{x_0}(ty)$ for some $y \in T_{x_0}M$ with $|y|_{g_{x_0}} = 1$; where

g_{x_0} and $|\cdot|_{g_{x_0}}$ denote the inner product and norm on $T_{x_0}M$ induced by the Riemannian metric g.

We notice that $O_G^{\gamma(t)}$ contains infinitely many elements for every $t>0$. Indeed, $O_G^{\gamma(t)}$ is a connected submanifold of M whose dimension is at least 1; if its dimension would be 0 for some $t_0>0$, by connectedness, $O_G^{\gamma(t_0)}$ would be a singleton, i.e.,

$$\gamma(t_0)\in \mathrm{Fix}_M(G)=\{x_0\}=\{\gamma(0)\},$$

which is a contradiction. Therefore, $\mathrm{card}\,O_G^{\gamma(t)}=+\infty$ for every $t>0$.

If for a fixed $t_0>0$, we choose different elements $\sigma_i\in G$, $i\in\mathbb{N}$ such that $\sigma_i\gamma(t_0)\in O_G^{\gamma(t_0)}$, then we also have $(\sigma_i\circ\gamma)(t)=\sigma_i\gamma(t)\in O_G^{\gamma(t)}$ for every $i\in\mathbb{N}$ and $t>0$; the latter statement immediately follows from the fact that $\sigma_i\in G$, $i\in\mathbb{N}$ are isometries, thus $t\mapsto(\sigma_i\circ\gamma)(t)$ are also geodesics of unit speed emanating from $x_0.$

Let us transplant the geodesic balls $B_g(\sigma_i\gamma(t);\rho)\subset M$, $i\in\mathbb{N}$, into the tangent space $T_{x_0}M$ by the exponential map $\exp_{x_0}$, i.e., $\exp_{x_0}^{-1}(B_g(\sigma_i\gamma(t);\rho))\subset T_{x_0}M$, $i\in\mathbb{N}$.

We claim that

$$\exp_{x_0}^{-1}(B_g(\sigma_i\gamma(t);\rho))\subseteq B^{x_0}(\exp_{x_0}^{-1}(\sigma_i\gamma(t)))=:B_i^t(\rho);\ i\in\mathbb{N}; \tag{4.13}$$

where $B^{x_0}(v)=\{z\in T_{x_0}M:|v-z|_{g_{x_0}}<\rho\}\subset T_{x_0}M$ for any $v\in T_{x_0}M.$ To see this, let $i\in\mathbb{N}$ and $t\in[0,\infty)$ be arbitrarily fixed. Take an element $z\in\exp_{x_0}^{-1}(B_g(\sigma_i\gamma(t);\rho))$, thus $\tilde z:=\exp_{x_0}(z)\in B_g(\sigma_i\gamma(t);\rho)$. If $z=\exp_{x_0}^{-1}(\sigma_i\gamma(t))$, we have nothing to prove. Otherwise, consider the geodesic triangle uniquely determined by the points x_0, $\sigma_i\gamma(t)$ and $\tilde z$, respectively. Since (M,g) is a Hadamard manifold, the Rauch comparison principle (see e.g. do Carmo [30, Proposition 2.5, p. 218]) implies that

$$|\exp_{x_0}^{-1}(\sigma_i\gamma(t))-z|_{g_{x_0}}=|\exp_{x_0}^{-1}(\sigma_i\gamma(t))-\exp_{x_0}^{-1}(\tilde z)|_{g_{x_0}}\le d_g(\sigma_i\gamma(t),\tilde z)<\rho;$$

which concludes the proof of (4.13). Since the geodesics $\sigma_i\circ\gamma$ are mutually different for any $i\in\mathbb{N}$, the angle between any two vectors $\exp_{x_0}^{-1}(\sigma_i\gamma(t))\in T_{x_0}M$ is positive and it does not depend on the value of $t>0$. Let $\theta_{ij}\in(0,\pi]$ be the angle between $v_i:=\exp_{x_0}^{-1}(\sigma_i\gamma(t))$ and $v_j:=\exp_{x_0}^{-1}(\sigma_j\gamma(t))$, $i\neq j$.

Geometrically, the semilines $\tau\mapsto\tau v_i\in T_{x_0}M$, $\tau>0$, move away in $T_{x_0}M$ from each other, independently of $t>0.$ Accordingly, it turns out that larger values of $t>0$ imply more mutually disjoint balls of the form $B_i^t(\rho)$. More precisely, if we define

$$\tilde m(t,\rho)=\sup\left\{k\in\mathbb{N}:B_i^t(\rho)\cap B_j^t(\rho)=\emptyset,\ \forall i\neq j\text{ with }i,j\in\{1,\dots,k\}\right\};$$

we claim that $\bar{m}(t;\rho) \to \infty$ as $t \to \infty$. To prove this, for every $k \geq 2$, let

$$t_k := \max\left\{\frac{\rho}{\sin\left(\frac{\theta_{ij}}{2}\right)} : i,j \in \{1,\dots,k\}, i \neq j\right\}.$$

Let $t_1 = 0$. By the latter definition, it turns out that $\bar{m}(t;\rho) \geq k$ whenever $t \geq t_k$. Let us observe that the sequence $(t_k)_k$ is nondecreasing and $\lim_{k\to\infty} t_k = +\infty$. The former statement is trivial, while the limit follows from the fact that the sequence of $w_i := \frac{v_i}{|v_i|_{g_{x_0}}}$, $i \in \mathbb{N}$ (belonging to the unit sphere of $T_{x_0}M$ with center $0 \in T_{x_0}M$) has a convergent subsequence, say $(w_{i_l})_l$; in particular, the sequence of angles $(\theta_{i_l i_{l+1}})_l$ converges to 0, which implies the validity of the required limit.

Now, let $(t_{k_l})_l$ be a strictly increasing subsequence of $(t_k)_k$ with $t_{k_1} = t_1 = 0$, and let $f : [0,\infty) \to [0,\infty)$ be defined by

$$f(s) = t_{k_l} + (s - l)(t_{k_{l+1}} - t_{k_l}),$$

for every $s \in [l, l+1)$, $l \in \mathbb{N}$. It is clear that f is strictly increasing and $\lim_{s\to\infty} f^{-1}(s) = +\infty$. By the above construction, for every $t > 0$, there exists a unique $l \in \mathbb{N}$ such that $t_{k_l} \leq t < t_{k_{l+1}}$.

In particular, it follows that $l = f^{-1}(t_{k_l}) \leq f^{-1}(t) < f^{-1}(t_{k_{l+1}}) = l + 1$, thus

$$f^{-1}(t) - 1 < l \leq k_l \leq \bar{m}(t;\rho).$$

The above relation immediately implies that $\bar{m}(t;\rho) \to \infty$ as $t \to \infty$.

On the other hand, by (4.13) and the fact that $\exp_{x_0}$ is a diffeomorphism, it turns out that

$$B_g(\gamma_i(t);\rho) \cap B_g(\gamma_j(t);\rho) = \emptyset,\ \forall i \neq j \text{ with } i,j \in \{1,\dots,\bar{m}(t;\rho)\}.$$

Therefore, we have that

$$m(\gamma(t);\rho) \geq \bar{m}(t;\rho), \tag{4.14}$$

and the aforementioned limit concludes the proof.

(iii) $\Rightarrow$ (ii) Let us assume that the set $\mathrm{Fix}_G(M)$ is not a singleton, i.e., there exists $x_0, x_1 \in \mathrm{Fix}_G(M)$ such that $\delta := d_g(x_0,x_1) > 0$. Since M is a Hadamard manifold, there exists a unique minimal geodesic $\gamma : \mathbb{R} \to M$, parametrized by arc-length, and passing through the points x_0 and x_1. Let $x_2 \in \mathrm{Im}\,\gamma \setminus \{x_0\}$ be such that $d_g(x_1,x_2) = \delta$ and $t_0 < t_1 < t_2$ with $x_i = \gamma(t_i)$, $i \in \{0,1,2\}$. Fix an arbitrary element $\sigma \in G$; in particular, $t \mapsto \tilde{\gamma}(t) := (\sigma \circ \gamma)(t)$ is also a geodesic.

It is clear that $\tilde{\gamma}(t_2) = \sigma x_2$ and due to the fact that $x_0, x_1 \in \mathrm{Fix}_G(M)$, it turns out that $\tilde{\gamma}(t_i) = \sigma x_i = x_i$, $i \in \{0,1\}$. Therefore, by the uniqueness of the geodesic between x_0 and x_1, it follows that $\tilde{\gamma}(t) = \gamma(t)$ for every $t \in [t_0,t_1]$. Since Riemannian manifolds are non-branching spaces, it follows in fact that $\tilde{\gamma} \equiv \gamma$, thus $\sigma x_2 = x_2$; since $\sigma \in G$ was arbitrary, we obtain that $x_2 \in \mathrm{Fix}_G(M)$ and $d_g(x_0,x_2) = d_g(x_0,x_1) + d_g(x_1,x_2) = 2\rho$. By repeating this argument, one can construct a sequence of points $(x_n)_n \subset M$ such that $x_n \in \mathrm{Fix}_G(M)$ and $d_g(x_0,x_n) = n\rho$, $n \in \mathbb{N}$. In particular, $d_g(x_0,x_n) \to \infty$ as $n \to \infty$ and, since $x_n \in \mathrm{Fix}_G(M)$ for every $n \in \mathbb{N}$, it follows that $m(x_n,\rho) = 1$, which is a contradiction.

<u>$(iii) \Rightarrow$ compact embeddings.</u> Based on the equivalence $(ii) \Leftrightarrow (iii)$ proved above, the compactness of the embeddings $W_G^{1,p}(M) \hookrightarrow L^q(M)$ in the admissible cases (S) and (MT) follow by Skrzypczak and Tintarev [111]. It remains to consider the admissible case (M), i.e., to prove the compactness of $W_G^{1,p}(M) \hookrightarrow L^\infty(M)$ whenever $n < p < \infty$.

To achieve this, we first claim that for every $\rho > 0$ fixed, one has

$$\inf_{y\in M} S(y,\rho)^{-1} > 0, \tag{4.15}$$

where for every $y \in M$ arbitrarily fixed, $S(y,\rho)$ is the embedding constant defined by the embedding

$$W_g^{1,p}(B_g(y,\rho)) \hookrightarrow C^0(\overline{B_g(y,\rho)});$$

i.e.,

$$S(y,\rho)^{-1} = \inf_{u\in W_g^{1,p}(B_g(y,\rho))\setminus\{0\}} \frac{\left(\int_{B_g(y,\rho)} |\nabla_g u|^p \mathrm{d}v_g + \int_{B_g(y,\rho)} |u|^p \mathrm{d}v_g\right)^{\frac{1}{p}}}{\sup_{x\in \overline{B_g(y,\rho)}} |u(x)|}, \tag{4.16}$$

see Lemma 4.3.2. Clearly, we have $S(y,\rho) > 0$. To prove (4.15), for $y \in M$ arbitrarily fixed, let $u \in W_g^{1,p}(B_g(y,\rho))\setminus\{0\}$ be a nonnegative function. By Lemma 4.3.1/(iii) it turns out that

$$\left(\int_{B_g(y,\rho)} |\nabla_g u|^p \,\mathrm{d}v_g\right)^{\frac{1}{p}} \geq \frac{C(n)}{n\,\omega_n^{\frac{1}{n}}} \left(\int_{B_e(0,\tilde\rho_y)} |\nabla u^*|^p \,\mathrm{d}x\right)^{\frac{1}{p}},$$

where $u^* : B_e(0,\tilde\rho_y) \to [0,\infty)$ denotes the Euclidean rearrangement of u; in particular, we have

$$\mathrm{Vol}_g(B_g(y,\rho)) = \mathrm{Vol}_e(B_e(0,\tilde\rho_y)) = \omega_n \tilde\rho_y^n \tag{4.17}$$

and

$$\sup_{x\in \overline{B_g(y,\rho)}} |u(x)| = \sup_{x\in \overline{B_e(0,\tilde\rho_y)}} |u^*(x)| = u^*(0).$$

On the other hand, by the Bishop-Gromov-type comparison principle (see Theorem 2.8.3) together with (4.17), one can see that $\rho \leq \tilde\rho_y$. Therefore, $B_e(0,\rho) \subseteq B_e(0,\tilde\rho_y)$ and

$W^{1,p}(B_e(0,\tilde\rho_y)) \subset W^{1,p}(B_e(0,\rho))$. Accordingly,

$$
\begin{aligned}
S(y,\rho)^{-1} &= \inf_{u\in W_g^{1,p}(B_g(y,\rho))\setminus\{0\}} \frac{\left(\int_{B_g(y,\rho)} |\nabla_g u|^p dv_g + \int_{B_g(y,\rho)} |u|^p dv_g\right)^{\frac{1}{p}}}{\sup\limits_{x\in \overline{B_g(y,\rho)}} |u(x)|} \\
&\ge \frac{C(n)}{n!^{\frac{1}{n}}} \inf_{\tilde u\in W^{1,p}(B_e(0,\tilde\rho_y))\setminus\{0\}} \frac{\left(\int_{B_e(0,\tilde\rho_y)} |\nabla \tilde u|^p dx + \int_{B_e(0,\tilde\rho_y)} |\tilde u|^p dx\right)^{\frac{1}{p}}}{\sup\limits_{x\in \overline{B_e(0,\tilde\rho_y)}} |\tilde u(x)|} \\
&\ge \frac{C(n)}{n!^{\frac{1}{n}}} \inf_{\tilde u\in W^{1,p}(B_e(0,\rho))\setminus\{0\}} \frac{\|\tilde u\|_{W^{1,p}(B_e(0,\rho))}}{\tilde u(0)} \\
&= \frac{C(n)}{n!^{\frac{1}{n}}} \inf_{\tilde u\in W^{1,p}(B_e(0,\rho))\setminus\{0\}} \frac{\|\tilde u\|_{W^{1,p}(B_e(0,\rho))}}{\sup\limits_{x\in\overline{B_e(0,\rho)}} |\tilde u(x)|} > 0.
\end{aligned}
$$

Since the last expression does not depend on $y\in M$, we conclude the proof of (4.15).

Now, let $(u_k)_{k\in\mathbb{N}} \subset W_G^{1,p}(M)$ be a bounded sequence and $\rho>0$ be an arbitrarily fixed number. Then, up to a subsequence, $u_k \rightharpoonup u$ in $W_G^{1,p}(M)$. Since G is a subgroup of $\mathrm{Isom}_g(M)$, for every $\tau_1,\tau_2\in G$, by a change of variables, one has

$$\|u_k - u\|_{W_g^{1,p}(B_g(\tau_1 y,\rho))} = \|u_k - u\|_{W_g^{1,p}(B_g(\tau_2 y,\rho))}.$$

Therefore, on account of the definition of $m(y,\rho)$ (see (4.4)), we have that

$$\|u_k - u\|_{W_g^{1,p}(B_g(y,\rho))} \le \frac{\|u_k - u\|_{W_g^{1,p}(M)}}{m(y,\rho)}.$$

By using Lemma 4.3.2 and the latter inequality, we obtain

$$\|u_k - u\|_{C^0(\overline{B_g(y,\rho)})} \le \frac{S(y,\rho)}{m(y,\rho)}\|u_k - u\|_{W_g^{1,p}(M)} \le \frac{S(y,\rho)}{m(y,\rho)}\left(\sup_{k\in\mathbb{N}}\|u_k\|_{W_g^{1,p}(M)} + \|u\|_{W_g^{1,p}(M)}\right).$$

According to (iii) and relation (4.15) we have that

$$\lim_{d_g(x_0,y)\to\infty} \frac{S(y,\rho)}{m(y,\rho)} = 0,$$

thus for every $\varepsilon>0$ there exists $R_\varepsilon>0$ such that

$$\sup_{d_g(x_0,y)\ge R_\varepsilon} \|u_k - u\|_{C^0(\overline{B_g(y,\rho)})} < \varepsilon, \quad \text{for every } k\in\mathbb{N}.$$

On the other hand, $u_k \rightharpoonup u$ in $W_G^{1,p}(M)$, thus by the Rellich-Kondrachov-type result (see

Lemma 4.3.2) it follows that $u_k \to u$ in $C^0\left(\overline{B_g(y;R_\varepsilon)}\right)$, hence there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\|u_k - u\|_{C^0\left(\overline{B_g(y;R_\varepsilon)}\right)} < \varepsilon, \quad \text{for all } k \geq k_\varepsilon.$$

The previous two inequalities yield that $u_k \to u$ in $L^\infty(M)$, which concludes the proof. □

Remark 4.3.1. (i) The quantity $m(x;\rho)$ from definition (4.4) can be easily estimated on nonpositively curved space forms. Indeed, for instance, if $n = 2$ and $G = O(2)$, $x_0 = 0$, then for $\rho > 0$ small enough, one has $m(x;\rho) \sim \frac{|x|}{\rho}$ as $|x| \to \infty$ in the Euclidean case $\mathbb{R}^2$, and $m(x;\rho) \sim \frac{|x|}{1-|x|^2}$ as $|x| \to 1$ in the Riemannian Poincaré disk $\mathbb{D} = \{x \in \mathbb{R}^2 : |x| < 1\}$ with constant sectional curvature -1, see Section 3.2.2.

(ii) Relation (4.14) can be viewed as a comparison of the maximal number of mutually disjoint geodesic balls with radius ρ on $(M;g)$ and on the Euclidean space $(\mathbb{R}^n;e)$, respectively. In fact, $\tilde{m}(t;\rho)$ is related to the particular inner product given by g_{x_0}, which is equivalent to the usual Euclidean metric e. This comparison result can be efficiently applied for every Hadamard manifold. In particular, in the usual n-dimensional Euclidean space, a simple covering argument shows that

$$\tilde{m}(t;\rho) = \omega\left(V_{\text{cap}}^{-1}(2\rho/t)\right) \quad \text{as} \quad t \to \infty;$$[1]

where $V_{\text{cap}}(r)$ denotes the area of the spherical cap of radius $r > 0$ on the unit $(n-1)$-dimensional sphere. For instance, when $n = 3$, we have

$$\tilde{m}(t;\rho) = \omega\left(\sin^{-2}(\rho/t)\right) \quad \text{as} \quad t \to \infty.$$

4.4 Compact Sobolev embeddings on Riemannian manifolds with bounded geometry

This section provides the counterpart of Theorem 4.3.3 in the case of Riemannian manifolds with bounded geometry:

Theorem 4.4.1. *Let $(M;g)$ be an n-dimensional Riemannian manifold with bounded geometry, and let G be a compact connected subgroup of* $\text{Isom}_g(M)$. *Then the following statements are equivalent:*

(i) G is coercive;

(ii) $(\mathbf{EC})_G$ *holds*;

(iii) the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ is compact for every *n-admissible pair* $(p;q)$;

(iv) the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ is compact for some *n-admissible pair* $(p;q)$.

[1] $f(t) = \omega(g(t))$ as $t \to \infty$ if there exist $c, \tau > 0$ such that $|f(t)| \geq c|g(t)|$ for every $t > \tau$.

In Theorem 4.4.1, the equivalence between condition (i) and the compactness of the embedding $W_G^{1,p}(M) \hookrightarrow L^q(M)$ for every n-admissible pair (p,q) in the case (S) is well-known by Tintarev [119, Theorem 7.10.12]; in addition, Górka [59] and Gaczkowski, Górka, and Pons [54] proved that a slightly stronger form of $(\mathbf{EC})_G$ implies (iii) in the admissible case (S) by using a Strauss-type argument. Thus, the novelty of Theorem 4.4.1 is the equivalence of the expansion condition $(\mathbf{EC})_G$ not only with the coerciveness of G but also with the validity of the compact embeddings in the full range of n-admissible pairs (p,q).

Proof of Theorem 4.4.1. $\underline{(i) \Longrightarrow (ii)}$ Let us assume by contradiction that $(\mathbf{EC})_G$ fails, i.e., there exist $K \in \mathbb{N}$ and a sequence $(x_k)_{k\in\mathbb{N}} \subset M$ such that

$$m(x_k,\varepsilon) \leq K \text{ for every } k \in \mathbb{N} \text{ and } d_g(x_0,x_k) \to \infty \text{ as } k \to \infty.$$

We are going to prove that $x_k \in O_{4(K+1)\varepsilon}$ for every $k \in \mathbb{N}$, which will imply in particular that $O_{4(K+1)\varepsilon}$ is unbounded, contrary to our assumption. We recall that $O_t = \{x \in M : \operatorname{diam} O_G^x \leq t\}$, $t > 0$.

In order to prove the claim, it suffices to show that $\operatorname{diam} O_G^{x_k} \leq 4(K+1)\varepsilon$ for every $k \in \mathbb{N}$. To do this, let $k \in \mathbb{N}$ be fixed and $m_k := m(x_k,\varepsilon) \leq K$. By the definition of $m(x_k,\varepsilon)$, there exist $\gamma_i := \gamma_i^k \in G$, $i \in \{1,\dots,m_k\}$, such that $B_g(\gamma_i x_k,\varepsilon) \cap B_g(\gamma_j x_k,\varepsilon) = \emptyset$, $\forall i \neq j$, $i,j \in \{1,\dots,m_k\}$, and the number $m_k \in \mathbb{N}$ is maximal with this property.

If we pick an arbitrary element $\gamma \in G$, it follows that there exists $i \in \{1,\dots,m_k\}$ such that $d_g(\gamma x_k,\gamma_i x_k) < 2\varepsilon$. If this is not the case, i.e., $d_g(\gamma x_k,\gamma_i x_k) \geq 2\varepsilon$ for every $i \in \{1,\dots,m_k\}$, it follows that $B_g(\gamma x_k,\varepsilon) \cap B_g(\gamma_i x_k,\varepsilon) = \emptyset$, $\forall i \in \{1,\dots,m_k\}$, i.e., one can find one more element $\gamma_{m_k+1} \in G$ with the disjointedness property, i.e., $B_g(\gamma_i x_k,\varepsilon) \cap B_g(\gamma_j x_k,\varepsilon) = \emptyset$, $\forall i \neq j$, $i,j \in \{1,\dots,m_k+1\}$, which contradicts the maximality of $m_k = m(x_k,\varepsilon)$. Accordingly,

$$\operatorname{diam} O_G^{x_k} \leq 4\varepsilon + \operatorname{diam}\{\gamma_i x_k : i \in \{1,\dots,m_k\}\}.$$

We claim that $\{\gamma_i x_k : i \in \{1,\dots,m_k\}\} \subset B_g(\gamma_1 x_k, 2m_k\varepsilon)$; clearly, we may put any element $\gamma_i \in G$, $i \in \{1,\dots,m_k\}$ instead of $\gamma_1 \in G$ in the right hand side of the above inclusion. We observe that for $m_k = 1$ the claim trivially holds. Thus, let $m_k \geq 2$. Assume the contrary, i.e., there exists $i_0 \in \{2,\dots,m_k\}$ such that $\gamma_{i_0} x_k \notin B_g(\gamma_1 x_k, 2m_k\varepsilon)$, that is

$$d_g(\gamma_{i_0} x_k, \gamma_1 x_k) \geq 2m_k\varepsilon.$$

We now fix a geodesic segment $\tilde{\gamma} : [0,1] \to O_G^{x_k}$ joining the points $\gamma_1 x_k \in O_G^{x_k}$ and $\gamma_{i_0} x_k \in O_G^{x_k}$; this can be done due to the fact that $O_G^{x_k}$ is a complete connected submanifold of (M,g) (as a closed submanifold of the complete Riemannian manifold (M,g)), see do Carmo [30, Corollary 2.10, p. 149]). Since $d_g(\tilde{\gamma}(0),\tilde{\gamma}(1)) = d_g(\gamma_1 x_k, \gamma_{i_0} x_k) \geq 2m_k\varepsilon$, by a continuity reason, we may fix $0 < t_1 < \dots < t_{m_k-1} < 1$ such that

$$d_g(\tau_1 x_k, \tilde\gamma(t_j)) = 2j\rho \quad \text{for every } j \in \{1, \dots, m_k - 1\}.$$

This particular choice clearly shows that $B_g(\tilde\gamma(t_j), \rho)$ are situated in some concentric annuli with the same width; more precisely,

$$B_g(\tilde\gamma(t_j), \rho) \subset B_g(\tau_1 x_k, (2j+1)\rho) \setminus B_g(\tau_1 x_k, (2j-1)\rho), \quad j \in \{1, \dots, m_k - 1\}.$$

Beside the above property, by $d_g(\tau_{i_0} x_k, \tau_1 x_k) \geq 2m_k\rho$, we also have that

$$B_g(\tilde\gamma(1), \rho) \cap B_g(\tau_1 x_k, (2m_k - 1)\rho) = \emptyset.$$

Combining all these constructions, it follows that the balls

$$B_g(\tilde\gamma(0), \rho) = B_g(\tau_1 x_k, \rho), B_g(\tilde\gamma(t_1), \rho), \dots, B_g(\tilde\gamma(t_{m_k-1}), \rho) \text{ and } B_g(\tilde\gamma(1), \rho) = B_g(\tau_{i_0} x_k, \rho)$$

are mutually disjoint sets, whose centers belong to $\mathrm{Im}\tilde\gamma \subset O_G^{x_k}$. Since the number of these balls is $m_k + 1$, this contradicts again the maximality of $m_k = m(x_k, \rho)$.

Accordingly,

$$\mathrm{diam} O_G^{x_k} \leq 4\rho + 4m_k\rho \leq 4(K+1)\rho,$$

which concludes the proof.

(ii) $\Longrightarrow$ (iii) We shall focus first on the Morrey case (M), i.e., we assume that $n < p < \infty$ and $q = \infty$; afterward we discuss the cases (S) and (MT).

Similarly to (4.15), we are going to prove that for every fixed $\rho > 0$ one has

$$\inf_{y \in M} S(y, \rho)^{-1} > 0, \tag{4.18}$$

where $S(y, \rho)$ is the embedding constant in $W_g^{1,p}(B_g(y, \rho)) \hookrightarrow C^0(\overline{B_g(y, \rho)})$, see (4.16).

Similarly to the proof of Lemma 4.3.2, we have that for any $\varepsilon > 0$ there exists a harmonic radius $r_H > 0$, such that for any $y \in M$, one can find a harmonic coordinate chart $\varphi : B_g(y, r_H) \to \mathbb{R}^n$, such that $\varphi(y) = 0$, and the components (g_{jl}) of g in this chart satisfy

$$\frac{1}{1+\varepsilon}\delta_{jl} \leq g_{jl} \leq (1+\varepsilon)\delta_{jl}$$

as bilinear forms. Fix $\rho < r_H$, then it is obvious that

$$B_e\left(0, \frac{\rho}{\sqrt{1+\varepsilon}}\right) \subset \Omega_y := \varphi(B_g(y, \rho)) \subset B_e(0, \sqrt{1+\varepsilon}\rho) \subset \mathbb{R}^n. \tag{4.19}$$

Combining (4.8) with (4.19), we have that

$$S(y;\rho)^{-1} = \inf_{u\in W_g^{1,p}(B_g(y;\rho))\setminus\{0\}} \frac{\left(\displaystyle\int_{B_g(y;\rho)} (|\nabla_g u|^p + |u|^p)\,dv_g\right)^{\frac{1}{p}}}{\displaystyle\sup_{x\in B_g(y;\rho)} |u(x)|}$$

$$\geq (1+\varepsilon)^{-\frac{n+p}{2p}} \inf_{u\in W_g^{1,p}(B_g(y;\rho))\setminus\{0\}} \frac{\left(\displaystyle\int_{\Omega_y} (|\nabla(u\circ\varphi^{-1})|^p + |u\circ\varphi^{-1}|^p)\,dx\right)^{\frac{1}{p}}}{\displaystyle\sup_{x\in\overline{\Omega_y}} |u\circ\varphi^{-1}(x)|}$$

$$\geq (1+\varepsilon)^{-\frac{n+p}{2p}} \inf_{f\in W^{1,p}(\Omega_y)\setminus\{0\}} \frac{\|f\|_{W^{1,p}(\Omega_y)}}{\|f\|_{C^0(\overline{\Omega_y})}}.$$

Let $f^*: \Omega_y^* \to [0,\infty)$ be the symmetric decreasing rearrangement of the function f (see Lieb and Loss [83, Section 3.3]), thus $\mathrm{Vol}_e(\Omega_y) = \mathrm{Vol}_e(\Omega_y^*)$ and

$$\inf_{f\in W^{1,p}(\Omega_y)\setminus\{0\}} \frac{\|f\|_{W^{1,p}(\Omega_y)}}{\|f\|_{C^0(\overline{\Omega_y})}} \geq \inf_{f^*\in W^{1,p}(\Omega_y^*)\setminus\{0\}} \frac{\|f^*\|_{W^{1,p}(\Omega_y^*)}}{\|f^*\|_{C^0(\overline{\Omega_y^*})}} = \inf_{f^*\in W^{1,p}(\Omega_y^*)\setminus\{0\}} \frac{\|f^*\|_{W^{1,p}(\Omega_y^*)}}{f^*(0)}.$$

Since $B_e\left(0;\frac{\rho}{\sqrt{1+\varepsilon}}\right) \subset \Omega_y^* \subset B_e(0;\sqrt{1+\varepsilon}\,\rho) \subset \mathbb{R}^n$, we have that

$$W^{1,p}\left(B_e\left(0;\frac{\rho}{\sqrt{1+\varepsilon}}\right)\right) \supset W^{1,p}(\Omega_y^*) \supset W^{1,p}(B_e(0;\sqrt{1+\varepsilon}\,\rho)).$$

Hence

$$\inf_{f^*\in W^{1,p}(\Omega_y^*)\setminus\{0\}} \frac{\|f^*\|_{W^{1,p}(\Omega_y^*)}}{f^*(0)} \geq \inf_{f^*\in W^{1,p}\left(B_e\left(0;\frac{\rho}{\sqrt{1+\varepsilon}}\right)\right)\setminus\{0\}} \frac{\|f^*\|_{W^{1,p}\left(B_e\left(0;\frac{\rho}{\sqrt{1+\varepsilon}}\right)\right)}}{\|f^*\|_{C^0\left(\overline{B_e\left(0;\frac{\rho}{\sqrt{1+\varepsilon}}\right)}\right)}} > 0,$$

meaning that $\inf\limits_{y\in M} S(y;\rho)^{-1} > 0$, which concludes the proof of (4.18).

Now, let $(u_k)_{k\in\mathbb{N}} \subset W_G^{1,p}(M)$ be a bounded sequence and $\rho > 0$ be an arbitrarily fixed number. Then, up to a subsequence, $u_k \rightharpoonup u$ in $W_G^{1,p}(M)$. Similarly to the proof of Theorem 4.3.3, we obtain

$$\|u_k - u\|_{C^0(\overline{B_g(y;\rho)})} \leq \frac{S(y;\rho)}{m(y;\rho)} \|u_k - u\|_{W_g^{1,p}(M)} \leq \frac{S(y;\rho)}{m(y;\rho)} \left(\sup_{k\in\mathbb{N}} \|u_k\|_{W_g^{1,p}(M)} + \|u\|_{W_g^{1,p}(M)}\right).$$

Due to the validity of $(\mathbf{EC})_G$ and relation (4.18), we have that

$$\lim_{d_g(x_0,y)\to\infty} \frac{S(y;\rho)}{m(y;\rho)} = 0,$$

thus for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\sup_{d_g(x_0,y)\geq R_\varepsilon} \|u_k - u\|_{C^0(\overline{B_g(y,\rho)})} < \varepsilon, \quad \text{for every } k \in \mathbb{N}.$$

Since $u_k \rightharpoonup u$ in $W_G^{1,p}(M)$, by the Rellich-Kondrachov-type result (see Lemma 4.3.2) it follows that $u_k \to u$ in $C^0\left(\overline{B_g(y,R_\varepsilon)}\right)$, hence there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\|u_k - u\|_{C^0(\overline{B_g(y,R_\varepsilon)})} < \varepsilon, \quad \text{for all } k \geq k_\varepsilon.$$

The previous two inequalities yield that $u_k \to u$ in $L^\infty(M)$, ending the proof in the admissible case $(\mathcal{M})$.

Now let us fix an arbitrary n-admissible pair (p,q) from $(\mathcal{S})$ or $(\mathcal{MT})$. A suitable modification of the above argument, based on Lemma 4.3.1/(ii), implies that

$$S(y,\rho)^{-1} := \inf_{u \in W_g^{1,p}(B_g(y,\rho))\setminus\{0\}} \frac{\left(\int_{B_g(y,\rho)} (|\nabla_g u|^p + |u|^p)\,dv_g\right)^{\frac{1}{p}}}{\left(\int_{B_g(y,\rho)} |u|^q\,dv_g\right)^{\frac{1}{q}}} > 0.$$

The latter inequality together with the validity of $(\mathbf{EC})_G$ implies that

$$\lim_{d_g(x_0,y)\to\infty} \frac{S(y,\rho)}{m(y,\rho)} = 0.$$

The rest is analogous as before, by using the Rellich-Kondrachov compactness result from Lemma 4.3.2.

$(iii) \Rightarrow (iv)$ Trivial.

$(iv) \Rightarrow (i)$ We follow the argument presented in Skrzypczak and Tintarev [112, Theorem 4.3]. In fact, the proof in the admissible case $(\mathcal{S})$ is given in Tintarev [119, Theorem 7.10.12]. Since the case $(\mathcal{MT})$ can be similarly discussed as $(\mathcal{S})$, we restrict our proof to the remaining admissible case $(\mathcal{M})$.

Suppose that G is not coercive, thus there exists a point $x_0 \in M$, a number $R > 0$ and a sequence $(x_k)_{k\in\mathbb{N}} \subset M$, such that $O_G^{x_k} \subset B_g(x_k,R)$ and $d_g(x_0,x_k) \to \infty$ as $k \to \infty$. Let $r \in (0, \mathrm{inj}_{(M,g)})$ and let us replace $(x_k)_k$ with a renumbered subsequence such that the distance between any two terms in the sequence will be greater than $2(R+r)$. We define a sequence of functions $(f_k)_{k\in\mathbb{N}}$ by

$$f_k(x) = \int_G (r - d_g(\tau x, x_k))_+ \,d\tau,$$

where the Haar measure of G is normalized to the value 1, and $u_+ = \max\{0,u\}$. It is easy to see that $f_k \in W_G^{1,p}(M)$ for every $k \in \mathbb{N}$ and any fixed $p \in (n,\infty)$. Indeed, since the support of f_k is a subset of $\overline{B_g(\sigma^{-1}x_k, r)}$ for all $k \in \mathbb{N}$, by an elementary computation using (2.3) and the volume-estimate (2.11), it follows that

$$\|f_k\|_{W_g^{1,p}(M)} \leq C(p,r,n),$$

where $C(p,r,n) > 0$ is independent of k. Therefore, $(f_k)_k$ is bounded in $W_G^{1,p}(M)$.

On the one hand, since the supports of the functions f_k are disjoint sets, we have that

$$\|f_l - f_k\|_{L^\infty(M)} = \|f_l\|_{L^\infty(M)} + \|f_k\|_{L^\infty(M)} \geq 2 \inf_{k \in \mathbb{N}} \|f_k\|_{L^\infty(M)}, \quad \forall l \neq k.$$

On the other hand,

$$\begin{aligned}
\mathrm{Vol}_g(B_g(x_k; R+r))\|f_k\|_{L^\infty(M)} \geq \int_M f_k(x)\,\mathrm{d}v_g &= \int_M \int_G (r - d_g(\sigma x, x_k))_+ \,\mathrm{d}\mu \,\mathrm{d}v_g(x) \\
&= \int_G \int_M (r - d_g(\sigma x, x_k))_+ \,\mathrm{d}v_g(x)\,\mathrm{d}\mu \\
&\overset{x:=\sigma^{-1}y}{=} \int_G \int_M (r - d_g(y, x_k))_+ \,\mathrm{d}v_g(y)\,\mathrm{d}\mu \\
&= \int_M (r - d_g(y, x_k))_+ \,\mathrm{d}v_g(y) \\
&\geq \frac{r}{2}\mathrm{Vol}_g\left(B_g\left(x_k, \frac{r}{2}\right)\right).
\end{aligned}$$

Since (M,g) is a Riemannian manifold with bounded geometry, then Vol_g is doubling on (M,g), thus

$$\|f_k\|_{L^\infty(M)} \geq C(r,R,n),$$

where $C(r,R,n) > 0$ does not depend on k. Thus $(f_k)_k$ is not a Cauchy sequence in $L^\infty(M)$, which is a contradiction. □

As a consequence of Theorem 4.4.1, we can prove the following corollary, which is related to the results obtained by Hebey and Vaugon [64] (see also Hebey [63, Theorems 9.5 & 9.6]):

Corollary 4.4.2. *Let (M,g) be a complete n-dimensional noncompact Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius, and let G be a compact connected subgroup of $\mathrm{Isom}_g(M)$ such that $\mathrm{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$. Assume that there exists $\eta = \eta(G,n) > 0$ such that for every $y \in M$ with $d_g(x_0,y) \geq 1$, one has*

$$H^l(O_G^y) \geq \eta\, d_g(x_0,y), \tag{H}$$

where $l = l(y) = \dim O_G^y \geq 1$ and H^l denotes the l-dimensional Hausdorff measure. Then the embedding $W_G^{1,p}(M) \hookrightarrow L^\infty(M)$ is compact for every $n < p < \infty$.

Proof. Let $y \in M$ be arbitrarily fixed such that $d_g(x_0,y) \geq 1$, and consider the elements $\sigma_i \in G$, $i = 1,\ldots,m(y,\rho)$ which appear in the definition of $m(y,\rho)$ in (4.4) for $\rho > 0$ small enough. Let also $l = l(y) = \dim O_G^y$. Notice that by the connectedness of G, we have $l \geq 1$. We claim that

$$H^l(O_G^y) \leq m(y;\rho) \sup_i H^l(B_g(\gamma_i y; k\rho) \cap O_G^y), \tag{4.20}$$

for every $k > 2$ independent of y. To see this, it is sufficient to prove that

$$O_G^y \subseteq \bigcup_i (B_g(\gamma_i y; k\rho) \cap O_G^y).$$

Let $x \in O_G^y$ be arbitrarily fixed. First, if $x \in \bigcup_i (B_g(\gamma_i y;\rho) \cap O_G^y)$, we have nothing to prove. If $x \notin \bigcup_i (B_g(\gamma_i y;\rho) \cap O_G^y)$, then there exists $i_0 \in \{1, \ldots, m(y;\rho)\}$ such that

$$d_g\left(x, \partial\left(B_g(\gamma_{i_0} y;\rho) \cap O_G^y\right)\right) < \rho.$$

Indeed, if the contrary holds, then $B_g(x;\rho) \cap B_g(\gamma_i y;\rho) = \emptyset$, $\forall i = 1, \ldots, m(y;\rho)$, thus $B_g(x;\rho)$ is a new ball in the definition of $m(y;\rho)$, contradicting the maximality of $m(y;\rho)$. Therefore, $d_g(x;\gamma_{i_0} y) < 2\rho$, which means that $x \in B_g(\gamma_{i_0} y; k\rho) \cap O_G^y$ for every $k > 2$, which proves (4.20).

We also notice that since $\mathrm{Fix}_M(G) = \{x_0\}$, one has that $O_G^y \subseteq \partial B_g(x_0; d_g(x_0;y))$. Indeed, if $x = \gamma y \in O_G^y$ then $d_g(x_0;x) = d_g(x_0;\gamma y) = d_g(\gamma^{-1} x_0;y) = d_g(x_0;y)$. Thus O_G^y is an l-dimensional submanifold of $\partial B_g(x_0; d_g(x_0;y))$, $l \leq n-1$. Therefore, a slight modification of Gallot, Hulin, and Lafontaine [56, Theorem 3.98] gives that for every $i = 1, \ldots, m(y;\rho)$,

$$H^l(B_g(\gamma_i y; k\rho) \cap O_G^y) \sim k^l \omega_l \rho^l (1 + o(\rho)) \text{ as } \rho \to 0,$$

whenever $k > 2$ is kept small (e.g., $k = 3$). To see this, we explore that $\exp_{\gamma_i y} : T_{\gamma_i y} M \to M$ is a local diffeomorphism at $0 \in T_{\gamma_i y} M$ with $d(\exp_{\gamma_i y})_0 = id$, while for small $\rho > 0$ one has $\exp^{-1}_{\gamma_i y}(B_g(\gamma_i y; k\rho) \cap O_G^y) = B_e(0; k\rho) \cap \exp^{-1}_{\gamma_i y}(O_G^y)$, and $0 < H^l(\exp^{-1}_{\gamma_i y}(O_G^y)) < \infty$.

Now, if we fix $\rho \in (0,1)$ from the usual range (see Gallot, Hulin, and Lafontaine [56]), it follows by (4.20) that

$$H^l(O_G^y) \leq m(y;\rho) k^{l+1} \omega_l \rho^l.$$

Hypothesis **(H)** and the latter estimate imply that $d_g(x_0;y) \leq m(y;\rho) k^{l+1} \omega_l \rho^l$. By using this inequality, one can obtain an estimate independent of $l = l(y)$, namely

$$d_g(x_0;y) \leq m(y;\rho) k^n \omega_{n-1}.$$

Letting $d_g(x_0;y) \to \infty$ immediately implies that $m(y;\rho) \to \infty$, thus the expansion condition $(\mathbf{EC})_G$ holds. Applying Theorem 4.4.1 concludes the proof. □

Let us provide two explicit examples where hypothesis **(H)** holds.

Example 4.4.1. Let $\mathrm{Sym}(n,\mathbb{R})$ be the set of symmetric $n \times n$ matrices with real values, $\mathrm{P}(n,\mathbb{R}) \subset \mathrm{Sym}(n,\mathbb{R})$ be the cone of symmetric positive definite matrices, and $\mathrm{P}(n,\mathbb{R})_1$ be the subspace of matrices in $\mathrm{P}(n,\mathbb{R})$ with determinant one. The set $\mathrm{P}(n,\mathbb{R})$ is endowed with the scalar product

$$\langle U,V\rangle_X = \mathrm{Tr}(X^{-1}VX^{-1}U) \quad \text{for all} \quad X \in \mathrm{P}(n,\mathbb{R}),\ U,V \in T_X(\mathrm{P}(n,\mathbb{R})) \simeq \mathrm{Sym}(n,\mathbb{R}),$$

where $\mathrm{Tr}(Y)$ denotes the trace of $Y \in \mathrm{Sym}(n,\mathbb{R})$. One can prove that $(\mathrm{P}(n,\mathbb{R})_1,\langle\,,\rangle)$ is a Riemannian manifold (with non-constant sectional curvature). On the other hand, since the scalar curvature of the Riemannian manifold $(\mathrm{P}(n,\mathbb{R})_1,\langle\,,\rangle)$ is constant, more precisely, $\mathrm{S} = -\frac{1}{8}n(n-1)(n+2)$, see Andai [8] and Moakher and Zéraï [91], it follows that its Ricci curvature is bounded from below.

The special linear group $SL(n)$ leaves $\mathrm{P}(n,\mathbb{R})_1$ invariant and acts transitively on it. Moreover, for every $\sigma \in SL(n)$, the map $[\sigma] : \mathrm{P}(n,\mathbb{R})_1 \to \mathrm{P}(n,\mathbb{R})_1$ defined by $[\sigma](X) = \sigma X \sigma^T$ is an isometry, where σ^T denotes the transpose of $\sigma.$ If $G = SO(n)$, we can prove that $\mathrm{Fix}_{\mathrm{P}(n,\mathbb{R})_1}(G) = \{I_n\}$, where I_n is the identity matrix; for more details, see Kristály [76]. On the other hand, the metric function on $\mathrm{P}(n,\mathbb{R})$ is given by $d_P(X,Y) = \sqrt{\mathrm{Tr}\left(\ln^2\left(X^{-\frac{1}{2}}YX^{-\frac{1}{2}}\right)\right)}$; see Kristály [74].

For simplicity, fix $n = 2$, and consider the following positive definite symmetric matrix

$$X = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{where} \quad a,c > 0 \quad \text{and} \quad ac - b^2 = 1.$$

Thus

$$O_G^X = \left\{\sigma X \sigma^T : \sigma = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},\ \theta \in [0,2\pi]\right\}.$$

One can see that

$$\mathcal{H}^1\left(O_G^X\right) = 2\pi\sqrt{a^2+2b^2+c^2} \ \text{ and } \ d_P(I_2,X) = \sqrt{\mathrm{Tr}\left(\ln^2(X)\right)} = \sqrt{\ln^2(\lambda_1)+\ln^2(\lambda_2)},$$

where λ_1 and λ_2 are the positive eigenvalues of the matrix X. Since $\sqrt{a^2+2b^2+c^2} = \sqrt{\lambda_1^2+\lambda_2^2}$, by using a Bernoulli-type inequality, it turns out that $\mathcal{H}^1\left(O_G^X\right) \geq \kappa_0 d_P(I_2,X)$, with $\kappa_0 := 2\pi$, which proves the validity of **(H)**.

Example 4.4.2. Let $G = O(d_1) \times \cdots \times O(d_k)$ with $d_i \geq 2$, $i = 1,\dots,k$, and $d_1 + \cdots + d_k = d$. Let $y = (y_1,\dots,y_k) \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_k}$. It is clear that $O_G^y = S^{d_1-1}_{|y_1|} \times \cdots \times S^{d_k-1}_{|y_k|}$, where S_r^{l-1} denotes the sphere with radius $r > 0$ in $\mathbb{R}^l$. Let $I(y) = \{i \in \{1,\dots,k\} : |y_i| \neq 0\}$.

Then $l = l(y) = \sum_{i\in I(y)} (d_i - 1)$ and

$$H^l(O_G^y) = \sum_{i\in I(y)} H^{d_i-1}(S_1^{d_i-1})|y_i|^{d_i-1} \geq 2\sum_{i\in I(y)} |y_i|^{d_i-1} = 2\sum_{i=1}^{k}|y_i|^{d_i-1}.$$

Now, let $|y_1| + \cdots + |y_k| = c \geq 1$. By the scaling $y_i := cz_i$, one has $|z_1| + \cdots + |z_k| = 1$ and

$$\sum_{i=1}^{k}|y_i|^{d_i-1} \geq c\sum_{i=1}^{k}|z_i|^{d_i-1}.$$

Note that the continuous function $(z_1,\dots,z_k) \mapsto \sum_{i=1}^{k}|z_i|^{d_i-1}$ attains its minimum on the simplex $|z_1| + \cdots + |z_k| = 1$, and this minimum is strictly positive, say $m_G > 0$ (otherwise, if $m_G = 0$, we would have all variables equal to zero, which is a contradiction). Summing up, it follows that

$$H^l(O_G^y) \geq 2cm_G = 2m_G(|y_1| + \cdots + |y_k|) \geq 2m_G|y|;$$

thus G satisfies the condition **(H)**.

4.5 Compact Sobolev embeddings on Randers spaces

At this point, the next natural step of generalization is the study of similar compact embedding results on noncompact Finsler manifolds. It turns out that in non-Riemannian Finsler settings the phenomena concerning Sobolev spaces may change dramatically. For instance, there exist noncompact Finsler-Hadamard manifolds (M,F) such that the Sobolev space $W_F^{1,2}(M)$ over (M,F) does not even admit a vector space structure, see Farkas, Kristály, and Varga [50], Kristály and Rudas [81] and Section 2.7. Therefore, the extension of Theorems 4.3.3 and 4.4.1 in the general Finsler case is not possible.

Despite such counterexamples, it can be proved that similar compactness results to Theorems 4.3.3 & 4.4.1 can be established on Randers spaces having finite reversibility constant (see Section 3.1).

In the following let (M,F) be an n-dimensional Randers space and $\mathrm{Isom}_F(M)$ be the isometry group of (M,F). In this case, $\mathrm{Isom}_F(M)$ is a closed subgroup of the isometry group of the underlying Riemannian manifold (M,g), see Deng [40, Proposition 7.1]. If G is a subgroup of $\mathrm{Isom}_F(M)$, then $W_{F,G}^{1,p}(M)$ stands for the subspace of G-invariant Sobolev functions of $W_F^{1,p}(M)$, i.e.,

$$W_{F,G}^{1,p}(M) = \left\{u \in W_F^{1,p}(M) : u\circ\xi = u \text{ for all } \xi \in G\right\}.$$

Finally, for any $y \in M$, let $m_F(y, \varepsilon)$ denote the maximal number of mutually disjoint geodesic Finsler balls with radius ε on the orbit $\mathcal{O}_G^y$. Then, one has the following embedding theorem.

Theorem 4.5.1. *Let (M, F) be an n-dimensional Randers space endowed with the Finsler metric $F : TM \to \mathbb{R}$,*

$$F(x, v) = \sqrt{g_x(v, v)} + \beta_x(v), \quad \forall (x, v) \in TM, \tag{4.21}$$

such that (M, g) is either a Cartan-Hadamard manifold or a Riemannian manifold with bounded geometry. Suppose that $\sup_{x \in M} |\beta_x|_g < 1$. Then the following results hold:

(i) For every n-admissible pair (p, q) the embedding $W_F^{1,p}(M) \hookrightarrow L^q(M)$ is continuous.

(ii) Let G be a compact connected subgroup of $\mathrm{Isom}_F(M)$ such that $m_F(y, \varepsilon) \to \infty$ as $d_F(x_0, y) \to \infty$ for some $x_0 \in M$ and $\varepsilon > 0$. Then the embedding $W_{F,G}^{1,p}(M) \hookrightarrow L^q(M)$ is compact for any n-admissible pair (p, q).

Proof. (i) For the sake of brevity let us introduce the notation

$$a := \sup_{x \in M} |\beta_x|_g < 1.$$

Recall that the volume form on the Randers space (M, F) is given by (3.3), Section 3.1. Therefore, one has that

$$(1 - a^2)^{\frac{n+1}{2}} \mathrm{d}v_g \leq \mathrm{d}v_F(x) \leq \mathrm{d}v_g. \tag{4.22}$$

Next, by using the definition of the polar transform of F, see (3.2), we obtain that

$$\begin{aligned} F^*(x, \xi) &\leq \frac{\sqrt{|\beta_x|_g^2 |\xi_x|_g^2 + (1 - |\beta_x|_g^2)|\xi_x|_g^2} + |\beta_x|_g |\xi_x|_g}{1 - |\beta_x|_g^2} \\ &= \frac{|\xi_x|_g(1 + |\beta_x|_g)}{1 - |\beta_x|_g^2} = \frac{|\xi_x|_g}{1 - |\beta_x|_g} \leq \frac{|\xi_x|_g}{1 - a}, \quad \forall (x, \xi) \in T^*M. \end{aligned} \tag{4.23}$$

On the other hand,

$$\begin{aligned} F^*(x, \xi) &= \frac{|\xi_x|_g^2}{\sqrt{g_x^{*2}(\beta_x, \xi_x) + (1 - |\beta_x|_g^2)|\xi_x|_g^2} + g_x(\beta_x, \xi_x)} \\ &\geq \frac{|\xi_x|_g^2}{\sqrt{|\beta_x|_g^2 |\xi_x|_g^2 + (1 - |\beta_x|_g^2)|\xi_x|_g^2} + |\beta_x|_g |\xi_x|_g} \\ &= \frac{|\xi_x|_g}{1 + |\beta_x|_g} \geq \frac{|\xi_x|_g}{1 + a}, \quad \forall (x, \xi) \in T^*M. \end{aligned} \tag{4.24}$$

Combining (4.22), (4.23) and (4.24), it follows that for every function $u \in W^{1,p}_F(M)$, we have

$$\frac{(1-a^2)^{\frac{n+1}{2}}}{(1+a)^p}\|u\|^p_{W^{1,p}_g(M)} \le \|u\|^p_{W^{1,p}_F(M)} \le \frac{1}{(1-a)^p}\|u\|^p_{W^{1,p}_g(M)}, \tag{4.25}$$

i.e., the Sobolev norms $\|\cdot\|_{W^{1,p}_F(M)}$ and $\|\cdot\|_{W^{1,p}_g(M)}$ are equivalent. Therefore, based on the continuous embeddings on the Riemannian manifold (M,g) (see Theorems and 4.2.2 and 4.2.3), we obtain that there exists a constant $C_{n,p} > 0$ such that

$$\|u\|_{L^q(M)} \le C_{n,p}\frac{1+a}{(1-a^2)^{\frac{n+1}{2p}}}\|u\|_{W^{1,p}_F(M)},$$

for all $u \in W^{1,p}_F(M)$ and for any n-admissible pair (p,q).

(ii) First of all, according to Deng [40, Proposition 7.1], G is a closed subgroup of the isometry group $\mathrm{Isom}_g(M)$ of the Riemannian manifold (M,g). Secondly, since

$$(1-a)d_g(x_0,y) \le d_F(x_0,y) \le (1+a)d_g(x_0,y),$$

the expansion condition $m_F(y,\rho) \to \infty$ as $d_F(x_0,y) \to \infty$ implies that $m\left(y,\frac{\rho}{1+a}\right) \to \infty$ as $d_g(x_0,y) \to \infty$. This means that condition $(\mathbf{EC})_G$ holds on the Riemannian manifold (M,g), which, in turn, yields the validity of the compact embeddings $W^{1,p}_G(M) \hookrightarrow L^q(M)$ by Theorems 4.3.3 & 4.4.1.

Now, let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}_{F,G}(M)$. From (4.25), it follows that $(u_k)_k$ is bounded in $W^{1,p}_G(M)$, thus, by the aforementioned compactness results, there exists a subsequence $(u_{k_l})_l$ which converges strongly to a function u in $L^q(M)$, for any n-admissible pair (p,q). This completes the proof. □

Note that the assumption $\sup_{x\in M}|\beta_x|_g < 1$ in Theorem 4.5.1 is equivalent to the finiteness of the reversibility constant of (M,F) (see (3.4), Section 3.1). Let us emphasize that this condition is indispensable for the validity of the continuous Sobolev embeddings $W^{1,p}_F(M) \hookrightarrow L^q(M)$. Indeed, the following example demonstrates that the continuous (and therefore, compact) Sobolev embeddings do not necessarily hold on Randers spaces having infinite reversibility constant.

Example 4.5.1. Let $n \ge 2$ and $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the n-dimensional Euclidean open unit ball. Consider the Funk metric $F : \mathbb{B}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$F(x,v) = \frac{\sqrt{(1-|x|^2)|v|^2 + \langle x,v\rangle^2}}{1-|x|^2} + \frac{\langle x,v\rangle}{1-|x|^2},$$

which defines the n-dimensional Finslerian Funk model $(\mathbb{B}^n,F)$, see Cheng and Shen [34, Example 2.1.2], and Shen [109, Example 1.3.4]. In fact, in the particular case $n = 2$ we recover the 2-dimensional Funk model $(\mathbb{D},F_F)$ presented in Section 3.2.1.

Recall that $(\mathbb{B}^n, F)$ is a noncompact Randers space with constant negative flag curvature $-\frac{1}{4}$, i.e., a Finsler-Hadamard manifold. Furthermore, the underlying Riemannian manifold is the Beltrami-Klein model having constant negative sectional curvature -1, which is a Cartan-Hadamard manifold. Nevertheless, the reversibility constant of $(\mathbb{B}^n, F)$ is $r_F = +\infty$, see (3.9).

Regarding the distance function on the Funk model, we have that

$$d_F(0, x) = -\ln(1 - |x|),$$

for all $x \in \mathbb{B}^n$, see Cheng and Shen [34, Example 2.1.2].

Now, let (p, q) be any n-admissible pair and consider the function $u : \mathbb{B}^n \to \mathbb{R}$ defined by

$$u(x) = e^{\frac{d_F(0,x)}{t}}\left(1 - e^{-d_F(0,x)}\right) = \frac{|x|}{(1-|x|)^{\frac{1}{t}}},$$

where $t > 0$ is a parameter. A direct calculation yields that

$$Du(x) = \frac{1}{t}e^{\frac{d_F(0,x)}{t}}\left[1 + (t-1)e^{-d_F(0,x)}\right] Dd_F(0, x).$$

By applying (2.3), we obtain that $F^*(x, Dd_F(0, x)) = 1$ for a.e. $x \in \mathbb{B}^n$, thus

$$\begin{aligned}\|u\|^p_{W^{1,p}_F(\mathbb{B}^n)} &= \int_{\mathbb{B}^n} e^{\frac{p\, d_F(0,x)}{t}}\left(1 - e^{-d_F(0,x)}\right)^p dv_F(x) \\ &\quad + \left(\frac{1}{t}\right)^p \int_{\mathbb{B}^n} e^{\frac{p\, d_F(0,x)}{t}}\left[1 + (t-1)e^{-d_F(0,x)}\right]^p dv_F(x).\end{aligned}$$

Since $dv_F(x) = dx$ (see Kristály and Rudas [81, Section 2.2]), it follows that

$$\begin{aligned}\|u\|^p_{W^{1,p}_F(\mathbb{B}^n)} &= \omega_{n-1}\int_0^1 \frac{s^p}{(1-s)^{\frac{p}{t}}} s^{n-1} ds + \omega_{n-1}\left(\frac{1}{t}\right)^p \int_0^1 \frac{(t-(t-1)s)^p}{(1-s)^{\frac{p}{t}}} s^{n-1} ds \\ &\leq \omega_{n-1}\int_0^1 s^{n-1}(1-s)^{-\frac{p}{t}} ds + \omega_{n-1}\int_0^1 s^{p+n-1}(1-s)^{-\frac{p}{t}} ds \\ &= \omega_{n-1}\left[\mathrm{B}\left(n, 1-\frac{p}{t}\right) + \mathrm{B}\left(p+n, 1-\frac{p}{t}\right)\right],\end{aligned}$$

where B denotes the Beta function.[2]

In the (S) & (MT) admissible cases, we have that

$$\|u\|^q_{L^q(\mathbb{B}^n)} = \omega_{n-1}\mathrm{B}\left(q + n, 1 - \frac{q}{t}\right).$$

[2] The Beta function is a special function defined as $\mathrm{B}(\alpha, \beta) = \int_0^1 s^{\alpha-1}(1-s)^{\beta-1} ds$, for every $\alpha, \beta > 0$, see Rudin [106, Theorem 8.20]. Note that when $\alpha \leq 0$ or $\beta \leq 0$, the integral is divergent.

By choosing $t := \frac{p+q}{2}$, it turns out that

$$1 - \frac{p}{t} = 1 - \frac{2p}{p+q} > 0 \quad \text{and} \quad 1 - \frac{q}{t} = 1 - \frac{2q}{p+q} < 0.$$

Therefore, we obtain that $\|u\|_{W_F^{1,p}(\mathbb{B}^n)} < +\infty$, whereas $\|u\|_{L^q(\mathbb{B}^n)} = +\infty$, which means that $u \in W_F^{1,p}(\mathbb{B}^n) \setminus L^q(\mathbb{B}^n)$.

In the case (M), let $t := \frac{p^2}{n} > 1$. Since $p > n$, it follows that

$$1 - \frac{p}{t} = 1 - \frac{n}{p} > 0; \quad \text{thus} \quad \|u\|_{W_F^{1,p}(\mathbb{B}^n)} < +\infty.$$

However, it is clear that $\|u\|_{L^\infty(\mathbb{B}^n)} = +\infty$, hence $u \in W_F^{1,p}(\mathbb{B}^n) \setminus L^\infty(\mathbb{B}^n)$.

In conclusion, the space $W_F^{1,p}(\mathbb{B}^n)$ cannot be continuously embedded into $L^q(\mathbb{B}^n)$ for any n-admissible pair (p, q), thus no further compact embedding can be expected.

Example 4.5.1 demonstrates that the theory of Sobolev spaces on Finsler manifolds cannot be treated analogously to the Riemannian case. Indeed, although the Funk model is a Finsler-Hadamard manifold of Randers-type, none of the continuous Sobolev embeddings are valid on the space because the reversibility constant is infinite. Due to the isometry result proved in Theorem 3.3.1, it follows that these unexpected phenomena are all valid on the Finsler-Poincaré ball and the Finsler-Poincaré upper half plane, all being Finsler-Hadamard manifolds (see Chapter 3). This is in sharp contrast with the Riemannian case, see Theorem 4.2.2.

Moreover, one can see that the underlying problem is much deeper, since the Sobolev spaces $W_F^{1,p}(\mathbb{B}^n)$ defined on the Funk ball may not even be vector spaces, see Kristály and Rudas [81]. Considering the isometry result from Theorem 3.3.1, this is also in concordance with the counterexample provided in Farkas, Kristály, and Varga [50] regarding the non-vector space structure of $W_F^{1,2}(\mathbb{D})$ on the 2-dimensional Finsler-Poincaré disk $(\mathbb{D}, F_P)$.

Chapter 5

Sobolev-type inequalities with singular terms

This chapter concerns Hardy inequalities, which belong to the family of Sobolev-type inequalities having singular terms. Other important examples include the Rellich inequality, Hardy-Sobolev-Maz'ya inequality, Hardy-type inequalities with multiple singularities, as well as several interpolations and improvements such as the Hardy-Rellich, Caffarelli-Kohn-Nirenberg or Brezis-Marcus inequality, etc. Many of these inequalities play a central role in the study of elliptic PDEs with singular potentials, while others have fundamental implications in quantum mechanics. Due to their significance, these inequalities have been objects of intense study, see, e.g., Balinsky, Evans, and Lewis [13], Brezis and Marcus [23], Brezis and Vázquez [24], Caffarelli, Kohn, and Nirenberg [25], Ghoussoub and Moradifam [57], Maz'ya [88] and Rellich [103].

After a short overview on the classical Hardy inequality in $\mathbb{R}^n$, we revisit a condition obtained by D'Ambrosio and Dipierro [38], which turns out to be sufficient for the validity of several Hardy-type inequalities on complete Riemannian manifolds. Then, we extend these results to forward complete Finsler manifolds. We also review a class of Hardy inequalities available on Finsler-Hadamard manifolds which have finite reversibility constant. Finally, we investigate a Hardy inequality with multiple singularities in the Finslerian setting. This chapter summarizes the results of Mester, Peter, and Varga [4] and Mester and Kristály [2].

5.1 The classical Hardy inequality: a short overview

The classical Hardy inequality in $\mathbb{R}^n$ usually refers to the relation

$$\int_{\mathbb{R}^n} |\nabla u(x)|^p \mathrm{d}x \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n), \tag{5.1}$$

where $n \geq 2$, $p \in (1, n)$, and the constant $\left(\frac{n-p}{p}\right)^p$ is sharp and can only be attained by the zero function. The latter result has its origin in the work of Hardy [61], where he proved an initial, one-dimensional version of (5.1). For a detailed historical summary regarding the original form of the Hardy inequality and its related results, we refer to Kufner, Maligranda, and Persson [82] and Hardy, Littlewood, and Pólya [62].

Since then, a rich theory has been developed around Hardy's classical inequality, producing an abundance of refinements, generalizations and modifications, with multiple applications in mathematical analysis and quantum mechanics. For a comprehensive treatment of the subject in the euclidean setting see Balinsky, Evans, and Lewis [13].

Due to the recent advances in geometric analysis, extensive efforts have been made towards the generalization of these Hardy inequalities to curved spaces. The first significant result was achieved in this direction by Carron [31], where he gave a sufficient condition for proving weighted L^2-type Hardy inequalities on complete noncompact Riemannian manifolds. This was followed by a series of refinements and improvements considering Hardy inequalities on Riemannian manifolds, see, e.g., D'Ambrosio and Dipierro [38], Kombe and Özaydin [70, 69], Xia [125] and Yang, Su, and Kong [126]. Various results were also extended to Finsler manifolds, see Farkas, Kristály, and Varga [50], Kristály and Repovš [80], Yuan, Zhao, and Shen [129], and Zhao [132].

In both cases, it turns out that the studied phenomena significantly depend on the geometric properties of the ambient space. For example, in the case of a Riemannian manifold, the curvature of the underlying structure plays an essential role in the development of such inequalities. Moreover, in the Finslerian case, besides the influence of the curvature, one may also need to take into account the non-Riemannian nature of the Finsler structure, which can be measured by the so-called reversibility constant and uniformity constant.

Accordingly, the purpose of the present chapter is the study of different Hardy-type inequalities on Finsler manifolds, by exploring the geometric and technical conditions which enable (or, in some cases, inhibit) such investigations.

5.2 Weighted Hardy inequalities on Finsler manifolds

D'Ambrosio and Dipierro [38] established a sufficient criteria in order to prove certain weighted Hardy inequalities on a complete Riemannian manifold (M, g). Namely, if $\Omega \subset M$ is an open set, ρ is a nonnegative function on Ω and $p > 1$, then, by assuming that ρ is p-superharmonic on Ω in weak sense, one can obtain the following inequality:

$$\int_\Omega |\nabla_g u(x)|^p \, dv_g \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{\rho(x)^p} |\nabla_g \rho(x)|^p \, dv_g, \quad \forall u \in C_0^\infty(\Omega).$$

By expanding the technique applied by D'Ambrosio and Dipierro [38], we prove Hardy inequalities involving a weight function on forward complete, not necessarily reversible Finsler manifolds. In addition, we recover and complement some of the results derived by Zhao [132]. In order to avoid technicalities, we consider the case $p = 2$, obtaining L^2-type Hardy inequalities. The results may be extended to any $p > 1$ by applying appropriate changes to the proofs.

In the remainder of this chapter let $(M; F)$ be a forward complete n-dimensional Finsler manifold and let $\quad M$ be an open set.

We say that a function $\quad 2\ W^{1;2}_{\mathrm{loc}}(\ )$ is p-superharmonic ($p \quad 2$) on $\quad$ in weak sense if

$$\int F\ (x; D\ (x))^{p\ \ 2}\ \ D\,'(x)\big(r_F\ \ (x)\big)\ \mathrm{d}v_F(x) \quad 0;$$

for every nonnegative test function $'\ 2\ C^1_0(\ )$. By the divergence theorem (2.9), this in turn is equivalent with the fact that $\quad_{F;p} \quad 0$ on $\quad$ in weak sense. Note that for $p = 2$, we simply say that $\quad$ is superharmonic, meaning that $\quad_F \quad 0$ on $\quad$ in the distributional sense. It turns out that this superharmonicity condition provides a su□cient criteria in order to prove several weighted Hardy-type inequalities on $(M; F)$.

First we recall the following lemma, which will be a crucial tool in our further developments. For the proof we follow Zhao [132, Theorem 3.1]. In the remainder of this chapter we omit the parameter $x\ 2\ M$ for the sake of brevity.

Lemma 5.2.1. *Let $(M; F)$ be a forward complete n-dimensional Finsler manifold and let $\quad M$ be an open set. Let $X\ 2\ L^1_{\mathrm{loc}}(\ )$ be a vector field and $f_X\ 2\ L^1_{\mathrm{loc}}(\ )$ a nonnegative function such that the following properties hold:*

(i) $f_X \quad \mathrm{div} X;$

(ii) $\frac{F^2(X)}{f_X}\ 2\ L^1_{\mathrm{loc}}(\ ).$

Then we have

$$4\int \frac{F^2(x; X)}{f_X} F\ \ ^2(x; Du)\ \mathrm{d}v_F \qquad \int u^2 f_X\ \mathrm{d}v_F; \quad 8u\ 2\ C^1_0(\ ):$$

Proof. By applying relation (2.8) and the Hölder inequality, we obtain

$$\begin{aligned}
\int u^2 f_X\ \mathrm{d}v_F &\qquad \int u^2 \mathrm{div} X\ \mathrm{d}v_F = \int D(u^2)(X)\ \mathrm{d}v_F \\
&\quad 2\int juj\ \ jDu(X)j\ \mathrm{d}v_F \quad 2\int jujF(x; X)F\ \ (x; Du)\ \mathrm{d}v_F \\
&\quad 2\left(\int u^2 f_X\ \mathrm{d}v_F\right)^{\frac{1}{2}} \left(\int \frac{F^2(x; X)}{f_X} F\ \ ^2(x; Du)\ \mathrm{d}v_F\right)^{\frac{1}{2}};
\end{aligned}$$

which completes the proof. □

By the appropriate choice of the vector field X and the function f_X, we obtain the following weighted Hardy inequality. In particular, by assuming that the reversibility constant of $(M; F)$ is finite, we deduce the quantitative analogue of Zhao [132, Theorem 4.1].

Theorem 5.2.2. *Let $(M; F)$ be a forward complete Finsler manifold and let $\quad M$ be an open set. Let $\quad 2\ W^{1;2}_{\mathrm{loc}}(\ )$ be a nonnegative function and $\quad 2$ R a constant, such that*

(i) $(1 \quad) \ _F \quad 0$ *on* $\quad$ *in weak sense;*

(ii) $\frac{F^{\ 2}(D\ )}{2}; \quad 2\ L^1_{\mathrm{loc}}(\ )$*:*

If $\quad 1$*, then*

$$\int F^{\ 2}(x; Du)\, \mathrm{d}v_F \quad \frac{(1 \quad)^2}{4} \int \frac{u^2}{2} F^{\ 2}(x; D\ )\, \mathrm{d}v_F; \quad 8u\ 2\ C_0^1(\ );$$

whereas if $\quad > 1$ *and* $r_F < +1$*, then*

$$\int F^{\ 2}(x; Du)\, \mathrm{d}v_F \quad \frac{(1 \quad)^2}{4r_F^2} \int \frac{u^2}{2} F^{\ 2}(x; D\ )\, \mathrm{d}v_F; \quad 8u\ 2\ C_0^1(\ ):$$

Proof. The proof is based on Lemma 5.2.1. Notice that the case $\quad = 1$ is trivial.

Let $\quad 2\ (0; 1)$, $\quad = \quad + \quad > 0$ on $\quad$, and define the vector field X and the function f_X on $\quad$ as

$$X = (1 \quad)\frac{r_F}{1} \quad \text{and} \quad f_X = (1 \quad)^2 \frac{F^{\ 2}(D\ )}{2}: \tag{5.2}$$

Since $\quad 2\ L^1_{\mathrm{loc}}(\ )$, $\frac{1}{\ } \quad \frac{1}{\ }$ and $D \quad = D$, it follows that X and $f_X\ 2\ L^1_{\mathrm{loc}}(\ )$.

If $\quad < 1$, by direct calculation we obtain

$$\frac{F^2(x; X)}{f_X} = \frac{F^{\ 2}(x; (1 \quad)D\ )}{(1 \quad)^2 F^{\ 2}(x; D\ )} = \quad ; \tag{5.3}$$

whereas when $\quad > 1$ and $r_F < +1$, we can write

$$\frac{F^2(x; X)}{f_X} = \frac{F^{\ 2}(x; (1 \quad)D\ )}{(\quad 1)^2 F^{\ 2}(x; D\ )} = \frac{F^{\ 2}(x; \ (\quad 1)D\ )}{F^{\ 2}(x; (\quad 1)D\ )} \quad r_F^2 \quad : \tag{5.4}$$

In both cases it turns out that $\frac{F^2(X)}{f_X}\ 2\ L^1_{\mathrm{loc}}(\ )$.

It remains to prove that $f_X \quad \mathrm{div} X$ in weak sense, for which we proceed similarly to Zhao [132, Theorem 4.1]. We sketch the proof for completeness.

Let $'\ 2\ C_0^1(\ )$ be arbitrarily fixed with $' \quad 0$, $K = \mathrm{supp}\, ' \quad$, and let $U \quad M$ be an open set such that $K \quad U$ and $\overline{U} \quad$ is compact. Let $k\ 2$ N$; k > \quad$, and define $_k = \inf f \quad ; kg$.

Let us consider the function $\ln\ _k\ 2\ L^2_{\mathrm{loc}}(\ )$. Then we have that $\ln\ _k\ 2\ W^{1;2}_F(U)$, thus there exists a sequence $(\ _n)_{n2\mathrm{N}} \quad C^1(U)$ such that $\quad _n \quad k$, $_n\ !\ \ _k$ a.e. on and

$$\int_U j\ln\ _n \quad \ln\ _k j^2 \mathrm{d}v_F\ !\ 0; \quad \int_U F^{\ 2}\left(\frac{D\ _n}{_n} \quad \frac{D\ _k}{_k}\right) \mathrm{d}v_F\ !\ 0 \quad \text{as}\ n\ !\ 1: \tag{5.5}$$

By using the functions $_n = \frac{'}{\frac{1}{n}} \; 2 \; C_0^1(\;)$, $_n$ 0 as test functions in (i), we obtain

$$(1 \quad)^2 \int \frac{D \;\; _n(r_F \;)}{\frac{2}{n}} \, ' \, dv_F \qquad (1 \quad) \int \frac{D \, ' (r_F \;)}{\frac{1}{n}} \, dv_F: \tag{5.6}$$

Case 1. If < 1, letting $n \; ! \; 1$, then $k \; ! \; 1$ in (5.6) yields

$$(1 \quad)^2 \int \frac{F^{\;2}(x; D \quad)}{2} \, ' \, dv_F \qquad (1 \quad) \int D \, ' \left(\frac{r_F}{1} \right) dv_F; \tag{5.7}$$

i.e., f_X $\quad$ div$X$ in weak sense on $\;$. Applying Lemma 5.2.1 and relation (5.3), then letting $\; ! \; 0$ completes the proof.

Case 2. Suppose > 1. For the second integral of (5.6) we have

$$\left| \frac{D \, ' (r_F \;)}{\frac{1}{n}} \right| \qquad F \;\; (D \, ') \, F(r_F \;) \, k \quad ^1 \qquad C \, F(r_F \;) \, k \quad ^1 \; 2 \; L^1(U);$$

so we may apply the dominated convergence theorem.

For the first integral in inequality (5.6) we can write

$$\frac{D \;\; _n(r_F \;)}{n^2} = {}_n \frac{D \;\; _n}{n} \left(\frac{r_F}{n} \right):$$

By using the dominated convergence theorem and relation (5.5), it follows that

$$\frac{r_F}{n} \quad ! \quad \frac{r_F}{k} \quad \text{in } L^2(U)$$

and

$${}_n \, F \left(\frac{D \;\; _n}{n} \right) \qquad k \;\; F \left(\frac{D \;\; _n}{n} \right) \qquad ! \quad k \;\; F \left(\frac{D \;\; _k}{k} \right) \quad \text{in } L^2(U):$$

Hence the sequence $_n \frac{D \; _n}{n}$ is bounded in $L^2(U)$, so up to a subsequence, it is weakly convergent in $L^2(U)$. Because of the pointwise convergence

$${}_n \frac{D \;\; _n}{n} \quad ! \quad {}_k \frac{D \;\; _k}{k} \quad \text{a.e. on} \quad ;$$

it follows that the convergence holds in weak sense, too.

Thus, if we let $n \; ! \; 1$ in (5.6), eventually for a subsequence, we get

$$(1 \quad)^2 \int \frac{D \;\; _k \; (r_F \;)}{\frac{2}{k}} \, ' \, dv_F \qquad (1 \quad) \int \frac{D \, ' (r_F \;)}{\frac{1}{k}} \, dv_F: \tag{5.8}$$

In order to pass to the limit $k \to \infty$, for the latter integral we can write

$$\left|\frac{D\,'(r_F\;)}{\frac{1}{k}}\right| = \tfrac{2}{k}\,D\,'\left(\tfrac{\;\;2}{k}{}^{\frac{2}{2}}\,r_F\;\right) \quad (\quad)^{\frac{}{2}}F\;(D\,')F\left(\tfrac{\;\;2}{2}\,r_F\;\right)$$
$$C(\quad)^{\frac{}{2}}(\quad)^{\frac{\;\;2}{2}}F\,(r_F\;) \in L^1(U);$$

since $\;>0$ and $(\quad)^{\frac{\;\;2}{2}}F\,(r_F\;)$, $(\quad)^{\frac{}{2}} \in L^2(U)$, so we can use the dominated convergence theorem. For the first integral of relation (5.8) we have

$$\frac{D_{\;k}\,(r_F\;)}{\frac{2}{k}}\,' = \frac{D_{\;k}\,(r_F\;)}{\frac{2}{k}}\quad f \qquad kg\,' = \frac{D\quad(r_F\;)}{2}\quad f \qquad kg\,' = \frac{F^2(r_F\;)}{2}\quad f \qquad kg\,':$$

If $\;\in(1;2]$, the previous expression is dominated by the function $C\frac{F^2(r_F\;)}{2}\in L^1(U)$, so we can apply the dominated convergence theorem.

For $\;\in(2;\infty)$, the sequence of nonnegative functions $\left(\frac{F^2(r_F\;)}{2}\quad f \qquad kg\,'\right)_{k\in\mathbb{N}}$ is monotone increasing, so we may use the monotone convergence theorem.

Thus, by letting $k\to\infty$ in (5.8), we obtain

$$(1\qquad)^2\int\frac{D\quad(r_F\;)}{2}\,'\,\mathrm{d}v_F \qquad (1\qquad)\int\frac{D\,'(r_F\;)}{1}\mathrm{d}v_F;$$

which is equivalent to (5.7), since $D\quad(r_F\;) = F^{\;2}(D\quad)$ by (2.1). Applying Lemma 5.2.1 and inequality (5.4), we obtain that

$$4r_F^2\int\qquad F^{\;2}(x;Du)\,\mathrm{d}v_F \qquad (1\qquad)^2\int\frac{u^2}{2}F^{\;2}(x;D\quad)\,\mathrm{d}v_F;\quad 8u\in C_0^\infty(\quad):$$

Now we pass to the limit $\;\to 0$. For the former integral we can use Fatou's lemma, while for the latter integral we can write $\quad F^2(r_F u)\quad C(\;+1)\in L^1(U);$ so we can apply the dominated convergence theorem. □

On the one hand, in the particular case $\;=0$ Theorem 5.2.2 yields the following Hardy inequality.

Corollary 5.2.3. *Let $(M;F)$ be a forward complete Finsler manifold and let $\qquad M$ be an open set. If $\;\in W^{1;2}_{\mathrm{loc}}(\quad)$ is a nonnegative function such that $\;$ is superharmonic on $\;$ in weak sense, and $\frac{F^{\;2}(D\;)}{2}\in L^1_{\mathrm{loc}}(\quad)$, then*

$$\int F^{\;2}(x;Du)\,\mathrm{d}v_F \quad \frac{1}{4}\int\frac{u^2}{2}F^{\;2}(x;D\;)\,\mathrm{d}v_F;\quad 8u\in C_0^\infty(\quad):$$

On the other hand, by choosing $\;=2+q;q>\;\;1$, we obtain the following Caccioppoli inequality.

Corollary 5.2.4. *Let (M,F) be a complete Finsler manifold with $r_F < \infty$, and let $\Omega \subset M$ be an open set. If $\rho \in W^{1,2}_{\rm loc}(\Omega)$ is a nonnegative function such that $\Delta_F \rho \leq 0$ on Ω in weak sense, and $q > -1$ such that $\rho^q F^{*2}(D\rho)$ and $\rho^{2+q} \in L^1_{\rm loc}(\Omega)$, then*

$$\int_\Omega \rho^{2+q} F^{*2}(x, Du)\, dv_F \geq \frac{(1+q)^2}{4r_F^2} \int_\Omega u^2 \rho^q F^{*2}(x, D\rho)\, dv_F, \quad \forall u \in C_0^\infty(\Omega).$$

Finally, in the remainder of this section we establish Hardy inequalities on Finsler-Hadamard manifolds having finite reversibility constant, by defining the weight function in Theorem 5.2.2 with the help of the Finslerian distance function d_F. For this, let (M,F) be a Finsler-Hadamard manifold with $r_F < \infty$, and let $\mathbf{S}$ denote the mean covariation of (M,F). For an $x_0 \in M$ arbitrarily fixed point let us denote by $r : M \to \mathbb{R}$, $r(x) = d_F(x_0, x)$ the distance function from the point x_0 on M. Note that as (M,F) is a Finsler-Hadamard manifold, we have $\mathrm{Cut}(x_0) = \emptyset$.

By applying Theorem 5.2.2 to a weight function defined with the help of the distance function r, we obtain the following Hardy inequality, which can be considered the quantitative version of the result given by Zhao [132, Theorem 1.2].

Theorem 5.2.5. *Let (M,F) be an n-dimensional Finsler-Hadamard manifold with $n \geq 3$, $r_F < \infty$ and $\mathbf{S} = 0$. If $\alpha \in (-\infty, 1)$, then for every $u \in C_0^\infty(M)$ we have*

$$\int_M r^{\alpha(2-n)} F^{*2}(x, Du)\, dv_F \geq \frac{(n-2)^2(1-\alpha)^2}{4r_F^2} \int_M r^{\alpha(2-n)} \frac{u^2}{r^2}\, dv_F.$$

Proof. Let $\Omega = M \setminus \{x_0\}$ be an open set, and define $\rho = r^{2-n} : \Omega \to [0, \infty)$, where $n = \dim M \geq 3$. We shall apply Theorem 5.2.2 with the weight function ρ.

Clearly, we have $\rho(x) > 0$ for every $x \in \Omega$. By using the definition of the reversibility constant r_F and the eikonal equation (2.3), we obtain that

$$F^{*2}(D\rho) = (n-2)^2 r^{2-2n} F^{*2}(\pm Dr) \leq (n-2)^2 r_F^2 r^{2-2n} \in L^1_{\rm loc}(\Omega), \tag{5.9}$$

thus $\rho \in W^{1,2}_{\rm loc}(\Omega)$.

Applying relation (2.3) again yields

$$\begin{aligned} \Delta_F \rho &= (2-n)\,\mathrm{div}(r^{1-n} \nabla_F r) \\ &= (2-n)\left((1-n) r^{-n} Dr(\nabla_F r) + r^{1-n} \Delta_F r\right) \\ &= (2-n)\, r^{-n}(1-n + r\Delta_F r). \end{aligned}$$

By using the Laplacian comparison principle for the Finslerian distance function (see Theorem 2.8.1), it follows that $\Delta_F r \geq \frac{n-1}{r}$ on Ω, thus

$$(1-\alpha)\Delta_F \rho = -(n-2)(1-\alpha)\, r^{-n}(1-n + r\Delta_F r) \leq 0 \quad \text{on } \Omega.$$

Similarly to (5.9), one can prove that $\frac{F\ ^2(D\ \ )}{2}$ and $\ 2\ L^1_{\mathrm{loc}}(\ \ )$, thus we can apply Theorem 5.2.2, which yields

$$\int r^{\ (2\ \ n)}F^{\ 2}(x; Du)\ \mathrm{d}v_F \quad \frac{(n\quad 2)^2(1\qquad)^2}{4}\int r^{\ (2\ \ n)}\frac{u^2}{r^2}F^{\ 2}(x;\quad Dr)\ \mathrm{d}v_F;$$

for every $u\ 2\ C^1_0(\ \ )$. Finally, from relation (2.3) we obtain

$$F^{\ 2}(x;\quad Dr)\quad \frac{1}{r_F^2}F^{\ 2}(x; Dr) = \frac{1}{r_F^2}; \tag{5.10}$$

for every $x\ 2\ \ $, and, since the set fx_0g has null Lebesgue measure, the proof is complete. □

Note that by choosing $\ \ = 0$ in Theorem 5.2.5, we recover the Hardy inequality obtained by Farkas, Kristály, and Varga [50, Proposition 4.1]:

Corollary 5.2.6. *Let $(M; F)$ be an n-dimensional Finsler-Hadamard manifold with $n\quad 3$, $r_F < 1$ and $\mathbf{S} = 0$. Then*

$$\int_M F^{\ 2}(x; Du)\ \mathrm{d}v_F \quad \frac{(n\quad 2)^2}{4r_F^2}\int_M \frac{u^2}{r^2}\ \mathrm{d}v_F; \quad 8u\ 2\ C^1_0(M): \tag{5.11}$$

Remark 5.2.1. (i) Theorem 5.2.5 and Corollary 5.2.6 represent the Finslerian generalization of the classical Hardy inequality (5.1) for $p = 2$ on Finsler-Hadamard manifolds. One can see that the obtained relations strongly depend on the geometry of the Finsler structure, which is manifested by the assumption $\mathbf{S} = 0$ and the finite reversibility condition $r_F < 1$. Moreover, the reversibility constant turns out to be embedded in the constant of the previous Hardy inequalities.

(ii) If $(M; F)$ is a reversible Finsler-Hadamard manifold, i.e., $r_F = 1$, the constant $\frac{(n\quad 2)^2}{4}$ in (5.11) is sharp and never achieved, see Farkas, Kristály, and Varga [50]. On the other hand, note that if we let $r_F\ \ !\ \ 1$, inequality (5.11) becomes trivial. The sharpness of the constant $\frac{(n\quad 2)^2}{4r_F^2}$ in the general case $r_F > 1$ is an open question.

Finally, we present the following logarithmic Hardy inequality:

Theorem 5.2.7. *Let $(M; F)$ be an n-dimensional Finsler-Hadamard manifold with $n\quad 2$, $r_F < 1$ and $\mathbf{S} = 0$, and consider a fixed number $\ \ 2\ \mathrm{R}nf1g$. If $\ \ < 1$ define $\ \ := r^{\ \ 1}(0; 1)$, while if $\ \ > 1$ set $\ \ := r^{\ \ 1}(1; +1)$. Then we have*

$$\int j\ln rj\ F^{\ 2}(x; Du)\ \mathrm{d}v_F \quad \frac{(1\qquad)^2}{4r_F^2}\int j\ln rj\ \frac{u^2}{(r\ln r)^2}\ \mathrm{d}v_F;\ 8u\ 2\ C^1_0(\ \ ): \tag{5.12}$$

Proof. Let $\ \ = (\qquad 1)\ln r:\qquad !\ \mathrm{R}$. Clearly, in both cases $\ \ < 1$ and $\ \ > 1$ we have $\ \ > 0$ on $\ \ $. Moreover, similarly to the proof of Theorem 5.2.5, we can show that $\ \ 2\ W^{1;2}_{\mathrm{loc}}(\ \ )$ and $\frac{F\ ^2(D\ \ )}{2}$, $\ \ 2\ L^1_{\mathrm{loc}}(\ \ )$.

Using the eikonal relation (2.3) and the Laplace comparison theorem (Theorem 2.8.1), we obtain that

$$(1-\alpha)\Delta_F\rho = (\alpha-1)\mathrm{div}(\rho\nabla_F\rho) = (\alpha-1)^2\,\mathrm{div}\left(\frac{1}{r}\rho\nabla_F r\right)$$
$$= (\alpha-1)^2\left(-\frac{1}{r^2}+\frac{\Delta_F r}{r}\right) \geq (\alpha-1)^2\frac{n-2}{r^2} \geq 0 \text{ on } \Omega,$$

so we can apply Theorem 5.2.2. If $\alpha > 1$, relation (2.3) yields (5.12). If $\alpha < 1$, Theorem 5.2.2 implies

$$\int_\Omega(-\ln r)^\alpha F^{*2}(x,Du)\,dv_F \geq \frac{(1-\alpha)^2}{4}\int_\Omega(-\ln r)^\alpha\frac{u^2}{(r\ln r)^2}F^{*2}(x,-Dr)\,dv_F,$$

for every $u\in C_0^\infty(\Omega)$. Applying relation (5.10) completes the proof. □

In particular, setting $\alpha=0$ and $\Omega := r^{-1}([0,1))$, and noting that the set $\{x_0\}$ has null Lebesgue measure yields

$$\int_\Omega F^{*2}(x,Du)\,dv_F \geq \frac{1}{4r_F^2}\int_\Omega\frac{u^2}{(r\ln r)^2}\,dv_F, \quad \forall u\in C_0^\infty(\Omega).$$

5.3 Gagliardo-Nirenberg inequality and uncertainty principle on Finsler manifolds

In the following we present a generalization of Lemma 5.2.1, which induces a weighted Gagliardo-Nirenberg inequality and a Heisenberg-Pauli-Weyl uncertainty principle. In the remainder of this section let (M,F) be a forward complete Finsler manifold and $\Omega\subseteq M$ an open set.

Lemma 5.3.1. *Let $X\in L^1_{\mathrm{loc}}(\Omega)$ be a vector field on Ω and $f_X\in L^1_{\mathrm{loc}}(\Omega)$ a nonnegative function such that $f_X\leq\mathrm{div}X$ and $\frac{F^2(X)}{f_X}\in L^1_{\mathrm{loc}}(\Omega)$. Then we have*

$$\int_\Omega|u|^s F^q(x,X)\,dv_F$$
$$\leq 4^{\frac{1}{p}}\left(\int_\Omega\frac{F^2(x,X)}{f_X}F^{*2}(x,Du)\,dv_F\right)^{\frac{1}{p}}\left(\int_\Omega\frac{F^{qp'}(x,X)}{f_X^{p'-1}}|u|^{\frac{ps-2}{p-1}}\,dv_F\right)^{\frac{1}{p'}}, \qquad (5.13)$$

for every function $u\in C_0^\infty(\Omega)$ and every real numbers $q\in\mathbb{R}$, $s>0$ and $p,p'>1$ such that $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. By applying the Hölder inequality and Lemma 5.2.1, we get

$$
\begin{aligned}
\int |u|^s F^q(x;X)\, dv_F &= \int |u|^{\frac{2}{p}} f_X^{\frac{1}{p}}\, F^q(x;X)\, f_X^{-\frac{1}{p}} |u|^{s-\frac{2}{p}}\, dv_F \\
&\leq \left(\int |u|^2 f_X\, dv_F\right)^{\frac{1}{p}} \left(\int F^{qp'}(x;X)\, f_X^{-\frac{p'}{p}}\, |u|^{p'(s-\frac{2}{p})}\, dv_F\right)^{\frac{1}{p'}} \\
&\leq 4^{\frac{1}{p}} \left(\int \frac{F^2(x;X)}{f_X} F^{*2}(x;Du)\, dv_F\right)^{\frac{1}{p}} \left(\int \frac{F^{qp'}(x;X)}{f_X^{p'-1}} |u|^{\frac{ps-2}{p-1}}\, dv_F\right)^{\frac{1}{p'}},
\end{aligned}
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. □

Let us define the function $w : \Omega \to \mathbb{R}$, $w(x) = \sqrt{\frac{F(x;X)}{f_X(x)}}$.

Choosing $p = 1 + \frac{2}{tz}$, $t, z > 0$ in (5.13) yields

$$
\left(\int |u|^s F^q(x;X)\, dv_F\right)^{\frac{1}{s}} \leq 2^{\frac{q}{s}} \left(\int w^2 F^{*2}(x;Du)\, dv_F\right)^{\frac{r}{2}} \left(\int w^{tz} |u|^z\, dv_F\right)^{\frac{1-r}{z}}, \tag{5.14}
$$

for all $u \in C_0^\infty(\Omega)$, where

$$
\frac{1}{s} = \frac{r}{2} + \frac{1-r}{z}, \quad \frac{1}{q} = \frac{1}{2} + \frac{1}{tz}, \quad \text{and} \quad r = \frac{t}{1+t} \in (0,1),
$$

while setting $q = 0$ in (5.13) implies

$$
\int |u|^s dv_F \leq 4^{\frac{1}{p}} \left(\int w^2 F^{*2}(x;Du)\, dv_F\right)^{\frac{1}{p}} \left(\int \frac{1}{f_X^{p'-1}} |u|^{\frac{ps-2}{p-1}} dv_F\right)^{\frac{1}{p'}}, \tag{5.15}
$$

for every $u \in C_0^\infty(\Omega)$, where $s > 0$ and $p, p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$.

As before, the proper choice of X and f_X in relations (5.14) and (5.15) implies a Gagliardo-Nirenberg inequality and an uncertainty principle. More specifically, by defining X and f_X as in the proof of Theorem 5.2.2 (see relation (5.2)) and setting $\;\; = 0$, we obtain that $w^2 = 1$, thus inequalities (5.14) and (5.15) yield the following theorems.

Theorem 5.3.2. *Let $\rho \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a nonnegative function such that ρ is superharmonic on Ω in weak sense. If $q \in \mathbb{R}$, $s, z > 0$ and $r \in (0,1)$, then*

$$
\left(\int |u|^s \frac{F^{*q}(x;D\rho)}{\rho^q}\, dv_F\right)^{\frac{1}{s}} \leq 2^{\frac{q}{s}} \left(\int F^{*2}(x;Du)\, dv_F\right)^{\frac{r}{2}} \left(\int |u|^z\, dv_F\right)^{\frac{1-r}{z}}
$$

for every $u \in C_0^\infty(\Omega)$, where

$$
\frac{1}{s} = \frac{r}{2} + \frac{1-r}{z} \quad \text{and} \quad \frac{1}{q} = \frac{1}{2} + \frac{1-r}{rz}.
$$

Theorem 5.3.3. *Let 2 $W^{1;2}_{\mathrm{loc}}($ $)$ be a nonnegative function such that is superharmonic on in weak sense. Let $s > 0$ and $p; p^0 > 1$ such that $\frac{1}{p} + \frac{1}{p^0} = 1$. Then for every u 2 $C^1_0($ $)$ we have*

$$\int juj^s \, \mathrm{d}v_F \quad 4^{\frac{1}{p}} \left(\int F \ ^2(x; Du) \, \mathrm{d}v_F \right)^{\frac{1}{p}} \left(\int \frac{2(p^0 \quad 1)}{F \ ^{2(p^0 \quad 1)}(x; D \ )} juj^{\frac{ps \quad 2}{p \quad 1}} \, \mathrm{d}v_F \right)^{\frac{1}{p^0}} :$$

On the one hand, taking $q = 1$ and $s = 2$ in Theorem 5.3.2 yields $r = \frac{1}{2}$ and $z = 2$, thus we obtain the following weighted Gagliardo-Nirenberg inequality:

Corollary 5.3.4. *Let 2 $W^{1;2}_{\mathrm{loc}}($ $)$ be a nonnegative function such that is superharmonic on in weak sense. Then*

$$\int u^2 \frac{F \ (x; D \ )}{} \, \mathrm{d}v_F \quad 2 \left(\int F \ ^2(x; Du) \, \mathrm{d}v_F \right)^{\frac{1}{2}} \left(\int u^2 \, \mathrm{d}v_F \right)^{\frac{1}{2}} ; \ 8u \ 2 \ C^1_0(\ ):$$

On the other hand, setting $p = s = 2$ in Theorem 5.3.3 implies the following weighted Heisenberg-Pauli-Weyl uncertainty principle:

Corollary 5.3.5. *Let 2 $W^{1;2}_{\mathrm{loc}}($ $)$ be a nonnegative function such that is superharmonic on in weak sense. Then*

$$\int u^2 \, \mathrm{d}v_F \quad 2 \left(\int F \ ^2(x; Du) \, \mathrm{d}v_F \right)^{\frac{1}{2}} \left(\int u^2 \frac{2}{F \ ^2(x; D \ )} \, \mathrm{d}v_F \right)^{\frac{1}{2}} ; 8u \ 2 \ C^1_0(\ ):$$

Remark 5.3.1. (i) Note that when $(M; F) = (M; g)$ is a Riemannian manifold, the Finsler-Laplace operator $_F$ and the gradient r_F reduce to the Laplace-Beltrami operator $_g$ and the Riemannian gradient operator r_g. Furthermore, by the Riesz representation theorem, one can identify the tangent space T_xM with its dual space $T_x M$, and the Finsler metrics F and F reduce to the norm $j \quad j_g$ induced by the Riemannian metric g. Therefore, the results presented in Section 5.2 and 5.3 extend the functional inequalities obtained by D'Ambrosio and Dipierro [38] to the class of forward complete, not necessarily reversible Finsler manifolds.

(ii) Let $(M; F) = (\mathrm{R}^n; e)$ be the standard n-dimensional Euclidean space. If we define the weight function to be the Euclidean norm, i.e., $(x) = jxj$, then we have $jr \quad (x)j = 1$, $8x \ 2 \ \mathrm{R}^n \ n \ f0g$, hence Corollaries 5.3.4 and 5.3.5 coincide with the classical Gagliardo-Nirenberg inequality and the Heisenberg-Pauli-Weyl uncertainty principle in the Euclidean setting.

5.4 Bipolar Hardy inequality on Finsler manifolds

One of the most challenging directions of extension regarding the classical Hardy inequality (5.1) is the study of so-called multipolar Hardy inequalities. Such problems are motivated by their application in molecular physics, quantum cosmology and combustion models, see, Bosi, Dolbeault, and Esteban [21], Felli, Marchini, and Terracini [51], Guo, Han, and Niu [60] and references therein.

The optimal multipolar counterpart of the unipolar inequality (5.1) in the case $p = 2$ on the n-dimensional Euclidean space was given by Cazacu and Zuazua [32], namely

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{m^2} \sum_{1 \leq i<j \leq m} \int_{\mathbb{R}^n} \left| \frac{x-x_i}{|x-x_i|^2} - \frac{x-x_j}{|x-x_j|^2} \right|^2 u^2 \mathrm{d}x, \quad \forall u \in C_0^\infty(\mathbb{R}^n); \tag{5.16}$$

where $x_1, \ldots, x_m \in \mathbb{R}^n$ represent pairwise distinct poles, $m \geq 2$, $n \geq 3$, and the constant $\frac{(n-2)^2}{m^2}$ is sharp.

This result was extended to complete Riemannian manifolds by Faraci, Farkas, and Kristály [48], as follows. Let (M, g) be an n-dimensional complete Riemannian manifold with $n \geq 3$, and consider the set of pairwise distinct poles $\{x_1, \ldots, x_m\} \subset M$, $m \geq 2$. Then the following multipolar Hardy inequality holds:

$$\begin{aligned} \int_M |\nabla_g u|_g^2 \mathrm{d}v_g \geq{}& \frac{(n-2)^2}{m^2} \sum_{1 \leq i<j \leq m} \int_M \left| \frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j} \right|_g^2 u^2 \mathrm{d}v_g \\ &+ \frac{n-2}{m} \sum_{i=1}^{m} \int_M \frac{d_i \Delta_g d_i - (n-1)}{d_i^2} u^2 \mathrm{d}v_g, \quad \forall u \in C_0^\infty(M); \end{aligned} \tag{5.17}$$

where $d_i := d_g(x_i, \cdot)$ denotes the Riemannian distance from the pole $x_i \in M$, $i = \overline{1,m}$. In addition, the constant $\frac{(n-2)^2}{m^2}$ is sharp in the bipolar case, i.e., when $m = 2$.

Regarding the role of the last term in inequality (5.17), a few remarks are in order (for the full analysis see Faraci, Farkas, and Kristály [48]):

- if the Ricci curvature of the manifold satisfies $Ric_{(M,g)} \geq c_0(n-1)g$ for some $c_0 > 0$, then it can be proved that the term in question is negative. Therefore, the last expression suitably modifies the counterpart of the flat setting given by (5.16) in order to hold true on positively curved spaces.

- in the negatively curved case, by using the Laplace comparison principle given by Theorem 2.8.1, one can prove that the last term in (5.17) provides stronger inequality when stronger curvature assumption is given.

- if $(M, g) = (\mathbb{R}^n, e)$ is chosen to be the standard n-dimensional Euclidean space, then $d_i(x) = |x - x_i|$, for all $x \in \mathbb{R}^n$, where $|\cdot|$ denotes the Euclidean norm. Hence, the last expression in (5.17) vanishes, and we obtain (5.16).

In the spirit of these studies, the purpose of this section is to investigate multipolar Hardy inequalities on complete, not necessarily reversible Finsler manifolds. Once again, we arrive to the conclusion that the obtained results are strongly influenced by the geometric properties of the given Finsler structure, expressed in terms of the reversibility constant r_F and uniformity constant l_F.

Our first result reads as follows.

Theorem 5.4.1. *Let (M,F) be a complete n-dimensional Finsler manifold with $n \geq 3$ and $l_F > 0$, and consider the set of pairwise distinct poles $\{x_1,\ldots,x_m\} \subset M$, where $m \geq 2$. Then*

$$\left(2 - \frac{l_F^2}{r_F^2}\right)\int_M F^{*2}(Du)\,\mathrm{d}v_F \geq (l_F - 2)\frac{(n-2)^2}{m^2}\int_M F^{*2}\Big(\sum_{i=1}^m \frac{Dd_i}{d_i}\Big)u^2\,\mathrm{d}v_F + l_F\frac{n-2}{m}\int_M \mathrm{div}\Big(J^*\Big(\sum_{i=1}^m \frac{Dd_i}{d_i}\Big)\Big)u^2\,\mathrm{d}v_F \qquad (5.18)$$

holds for every nonnegative function $u \in C_0^\infty(M)$, where $d_i(x) := d_F(x,x_i)$ denotes the Finslerian distance from the point x to the pole x_i, $i = \overline{1,m}$.

Proof. First, let us remark that the condition $l_F > 0$ implies the fact that $r_F < \infty$, see Section 2.5. We start by deducing some relations which will be necessary for our arguments. For any $x \in M$ and $\alpha,\beta \in T_x^*M$, we have the following inequalities:

- by applying (2.2) for $t = \frac{1}{2}$, we obtain

$$F^{*2}(x,\alpha+\beta) \leq 2F^{*2}(x,\alpha) + 2F^{*2}(x,\beta) - l_F F^{*2}(x,\beta-\alpha). \qquad (5.19)$$

- due to the strict convexity of F^{*2}, one can derive that

$$F^{*2}(x,\beta-\alpha) \geq F^{*2}(x,\beta) - 2\alpha(J^*(x,\beta)) + l_F F^{*2}(x,-\alpha). \qquad (5.20)$$

- since $r_F < \infty$, we have

$$F^*(x,-\alpha) \geq \frac{F^*(x,\alpha)}{r_F}. \qquad (5.21)$$

Combining relations (5.19) – (5.21) yields

$$F^{*2}(x,\alpha+\beta) \leq \left(2 - \frac{l_F^2}{r_F^2}\right)F^{*2}(x,\alpha) + (2 - l_F)F^{*2}(x,\beta) + 2l_F\,\alpha(J^*(x,\beta)), \qquad (5.22)$$

for all $x \in M$ and $\alpha,\beta \in T_x^*M$.

Now consider the pairwise distinct poles $x_1,\ldots,x_m \in M$ where $m \geq 2$, and let $d_i := d_F(\cdot,x_i)$ denote the Finslerian distance to the pole x_i, $i = \overline{1,m}$. Also, let $u \in C_0^\infty(M)$ be a nonnegative function on M. Applying (5.22) with the choices

$$= Du \quad \text{and} \qquad = \frac{n \quad 2}{m} u \sum_{i=1}^{m} \frac{Dd_i}{d_i};$$

then integrating over M results in

$$\begin{aligned} 0 \quad & \int_M F^{\ 2}\Big(Du + \frac{n \quad 2}{m} u \sum_{i=1}^{m} \frac{Dd_i}{d_i}\Big) \mathrm{d}v_F \\ & \Big(2 \quad \frac{l_F^2}{r_F^2}\Big) \int_M F^{\ 2}(Du)\mathrm{d}v_F + (2 \quad l_F)\frac{(n \quad 2)^2}{m^2} \int_M F^{\ 2}\Big(\sum_{i=1}^{m} \frac{Dd_i}{d_i}\Big) u^2 \mathrm{d}v_F \\ & + l_F \frac{n \quad 2}{m} \int_M D(u^2)\Big(J \ \Big(\sum_{i=1}^{m} \frac{Dd_i}{d_i}\Big)\Big) \mathrm{d}v_F : \end{aligned}$$

Using the divergence theorem (2.4) completes the proof. □

Remark 5.4.1. Note that Theorem 5.4.1 represents indeed the Finslerian counterpart of the Riemannian inequality (5.17).

When $(M;F) = (M;g)$ is a Riemannian manifold, we have $l_F = r_F = 1$, while the operators r_F and $\ _F$ coincide with r_g and $\ _g$, respectively. Moreover, the tangent space T_xM and its dual space $T_x M$ can be identified, and the Finsler metrics F and F are in fact the norm $j \ j_g$ associated to the Riemannian metric g. Thus the Hardy inequality (5.18) reduces to the following expression:

$$\int_M j \, r_g u j_g^2 \mathrm{d}v_g \qquad \frac{(n \quad 2)^2}{m^2} \int_M \Big|\sum_{i=1}^{m} \frac{r_g d_i}{d_i}\Big|_g^2 u^2 \mathrm{d}v_g + \frac{n \quad 2}{m} \sum_{i=1}^{m} \int_M \mathrm{div}\Big(\frac{r_g d_i}{d_i}\Big) u^2 \mathrm{d}v_g : \tag{5.23}$$

Now we expand the first term of the right hand side. First of all, by using the eikonal equation (2.3), one has

$$\Big|\frac{r_g d_i}{d_i} \qquad \frac{r_g d_j}{d_j}\Big|_g^2 = \frac{1}{d_i^2} + \frac{1}{d_j^2} \qquad 2\frac{g(r_g d_i; r_g d_j)}{d_i d_j};$$

for every $i; j \ 2 \ f1; : : : ; mg$.

Then, using the 'expansion of the square' method and the eikonal equation again, we obtain

$$
\begin{aligned}
\left|\sum_{i=1}^{m} \frac{\nabla_g d_i}{d_i}\right|_g^2 &= \sum_{i,j=1}^{m} g\left(\frac{\nabla_g d_i}{d_i}, \frac{\nabla_g d_j}{d_j}\right) \\
&= \sum_{i=1}^{m} \frac{1}{d_i^2} + 2 \sum_{1 \le i<j \le m} \frac{g(\nabla_g d_i, \nabla_g d_j)}{d_i d_j} \\
&= m \sum_{i=1}^{m} \frac{1}{d_i^2} - \sum_{1 \le i<j \le m} \left|\frac{\nabla_g d_i}{d_i} - \frac{\nabla_g d_j}{d_j}\right|_g^2 .
\end{aligned}
$$

On the other hand, considering the second term of the right hand side of (5.23), we have

$$
\operatorname{div}\left(\frac{\nabla_g d_i}{d_i}\right) = \frac{d_i \Delta_g d_i - 1}{d_i^2}, \quad \text{for all } i = \overline{1,m}.
$$

Substituting the expressions above, then rearranging the terms yields that (5.23) is equivalent to relation (5.17).

Therefore, Theorem 5.4.1 extends the multipolar Hardy inequality obtained by Faraci, Farkas, and Kristály [48, Theorem 1.1] to the class of complete Finsler manifolds, provided that the uniformity constant l_F is nonzero.

Applying Theorem 5.4.1 by choosing $m = 2$ results in the following bipolar Hardy inequality:

Theorem 5.4.2. *Let (M, F) be a complete n-dimensional Finsler manifold with $n \ge 3$ and $l_F > 0$. If $x_1, x_2 \in M$, $x_1 \neq x_2$, then*

$$
\begin{aligned}
\int_M F^{*2}(Du)\,dv_F \ge\ & \frac{l_F(2-l_F)}{2-\left(\frac{l_F}{r_F}\right)^2} \frac{(n-2)^2}{4} \int_M F^{*2}\left(\frac{Dd_2}{d_2} - \frac{Dd_1}{d_1}\right) u^2\,dv_F \\
&+ \frac{l_F}{2-\left(\frac{l_F}{r_F}\right)^2} \frac{n-2}{2} \int_M \operatorname{div}\left(J^*\left(\frac{Dd_1}{d_1} + \frac{Dd_2}{d_2}\right)\right) u^2\,dv_F \\
&- \frac{2-l_F}{2-\left(\frac{l_F}{r_F}\right)^2} \frac{(n-2)^2}{2} \int_M \left(\frac{1}{d_1^2} + \frac{1}{d_2^2}\right) u^2\,dv_F
\end{aligned}
\tag{5.24}
$$

holds for every nonnegative function $u \in C_0^\infty(M)$.

Proof. Let $x_1, x_2 \in M$ be two distinct poles and $d_1, d_2 : M \to [0, \infty)$ the associated distance functions. By using (5.19) and the eikonal equation (2.3), we obtain

$$
F^{*2}\left(x, \frac{Dd_1}{d_1} + \frac{Dd_2}{d_2}\right) \le 2\left(\frac{1}{d_1^2} + \frac{1}{d_2^2}\right) - l_F F^{*2}\left(x, \frac{Dd_2}{d_2} - \frac{Dd_1}{d_1}\right),
\tag{5.25}
$$

for a.e. $x \in M$. Applying Theorem 5.4.1 in the case $m = 2$, then using the inequality above completes the proof . □

Remark 5.4.2. The proof of an expanded form of inequality (5.18) appears to be a di□cult problem to solve. This can be contributed to the fact that the Legendre transform J associated to the Finsler metric F is usually not linear. Furthermore, the 'expansion of the square' method cannot be applied due to the lack of an appropriate inner product. Therefore, the sensible approach is to use suitable estimates, such as inequality (2.2) or (5.25), but such approximations do not produce the desired results in the multipolar case. Nevertheless, to our knowledge Theorems 5.4.1 and 5.4.2 seem to be the first contributions considering multipolar Hardy inequalities in the Finslerian setting.

Regarding the role of the constants l_F and r_F in the previous inequalities, we remark the following example.

Example 5.4.1. Let $(\mathrm{B}^n; F)$ be the n-dimensional Euclidean open unit ball endowed with the Funk metric, see Section 3.2.1 or Example 4.5.1. In this case, we have that $r_F = +1$ and $l_F = 0$ (see relation (3.9)), thus both inequalities (5.18) and (5.24) reduce to trivial statements. This particular example indicates the importance of the condition $l_F > 0$ in Theorems 5.4.1 and 5.4.2.

Chapter 6

Application to partial differential equations

The primary application of functional inequalities manifests in the theory of partial differential equations. When studying different elliptic PDEs and the associated BVPs via variational methods, the appropriate Sobolev inequalities and embedding results provide a tool to analyze the energy functional associated with the given problem. This way, one can verify essential properties of the energy functional such as sequential lower semicontinuity or the Palais–Smale condition. These conditions in turn enable us to prove existence/uniqueness/multiplicity results by applying certain minimization and/or minimax arguments, see e.g., Willem [123].

Due to the unusual phenomena which can result from the anisotropic nature of the Finsler metric, the adaptation of the standard variational methods in the case of Finsler manifolds requires careful analysis and increased attention. Various elliptic PDEs associated with the Finsler-Laplace operator have been studied on Minkowski spaces, see Alvino et al. [6], Farkas, Fodor, and Kristály [49], Ferone and Kawohl [52], as well as on more general Finsler manifolds, see Farkas, Kristály, and Varga [50], Kristály and Rudas [81] and Ohta and Sturm [99].

Accordingly, this chapter offers a demonstration of the application of the Sobolev inequalities and embeddings proved in Chapters 4 & 5, by presenting a multiplicity result concerning an elliptic problem defined on a Randers space $(M; F)$. This chapter is based on Farkas, Kristály, and Mester [1].

6.1 Elements from the theory of calculus of variations

The theory of calculus of variations pertains to the optimization of functionals. Therefore, variational methods can be translated to many other fields of science, as several problems arising in applications can be formulated as a minimization/maximization problem. In the particular case of PDEs, it can be proved that a weak solution of an elliptic problem coincides with a minimum point of the so-called energy functional associated to the given problem. This idea enables us to treat the PDE from an entirely different perspective, by using variational approaches.

In the following, without the sake of completeness, we recall some fundamental variational principles that will be utilized in the applications. For a comprehensive treatment of the subject, see Kristály, Rădulescu, and Varga [79] or Willem [123].

Let $(X, \|\cdot\|)$ be a real Banach space and X^* the dual space of X. We start off with some definitions.

Definition 6.1.1. We say that a sequence $(u_n)_{n\in\mathbb{N}} \subset X$ converges weakly to $u \in X$ if $\lim_{n\to\infty} L(u_n) = L(u)$ for every linear functional $L \in X^*$. In this case, we use the notation $u_n \rightharpoonup u$ in X.

Definition 6.1.2. A functional $f : X \to \mathbb{R}$ is said to be sequentially weakly lower semicontinuous (s.w.l.s.c.) if for every weakly convergent sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $u_n \rightharpoonup u$ in X, one has $f(u) \le \liminf_{n\to\infty} f(u_n)$.

Definition 6.1.3. A functional $f : X \to \mathbb{R}$ is coercive if for every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ with $\lim_{n\to\infty} \|u_n\| = \infty$, it follows that $\lim_{n\to\infty} f(u_n) = \infty$.

Definition 6.1.4. A function $f \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (denoted by $(PS)_c$-condition) if every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ satisfying

$$\lim_{n\to\infty} f(u_n) = c \quad \text{and} \quad \lim_{n\to\infty} \|f'(u_n)\| = 0$$

has a convergent subsequence.

We say that $f \in C^1(X,\mathbb{R})$ satisfies the *Palais-Smale condition* (shortly, (PS)-condition) if it satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.

Combining the Palais-Smale condition with Ekeland's variational principle (see Ekeland [46]) one can obtain the following fundamental critical point result:

Theorem 6.1.1. (Willem [123, Corollary 2.5]) *Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^1(X,\mathbb{R})$ be bounded from below. If f satisfies the $(PS)_c$-condition at level $c = \inf_X f$, then c is a critical value of f, i.e., there exists a point $u \in X$ such that $f(u) = c$ and $f'(u) = 0$.*

Another key critical point result is the celebrated mountain pass theorem, whose original version can be formulated as follows.

Theorem 6.1.2. (Ambrosetti and Rabinowitz [7]) *Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^1(X,\mathbb{R})$. Suppose that*

$$\inf_{\|u - u_0\| = \rho} f(u) > \max\{f(u_0), f(u_1)\}$$

for some $\rho \in \mathbb{R}$ *and* $u_0 \neq u_1 \in X$ *with* $0 < \rho < \|u_0 - u_1\|$*. If* f *satisfies the* $(PS)_c$*-condition at level*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = u_0 \text{ and } \gamma(1) = u_1\},$$

then c *is a critical value of* f *with* $c \geq \beta$*.*

The next theorems concern multiplicity results à la Ricceri [104, 105]. We first remark the well-known three critical points theorem of Pucci and Serrin [100], which is in fact a consequence of a mountain pass-type result.

Theorem 6.1.3. (Pucci and Serrin [100]) *Let* $(X, \|\cdot\|)$ *be a Banach space. If a function* $f \in C^1(X, \mathbb{R})$ *satisfies the* (PS)*-condition and it has two different local minimum points, then* f *has at least three distinct critical points.*

Regarding the stability of the previous three critical points, the following result can be formulated.

Theorem 6.1.4. (Ricceri [105]) *Let* $(X, \|\cdot\|)$ *be a separable and reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* Φ *is sequentially weakly lower semi-continuous and its Gâteaux derivative admits a continuous inverse on* X^**, while the Gâteaux derivative of* Ψ *is compact. Let* $I \subseteq \mathbb{R}$ *be an interval such that*

$$\lim_{\|u\| \to \infty} (\Phi(u) + \lambda \Psi(u)) = +\infty$$

for all $\lambda \in I$*. Furthermore, assume that there exists a continuous concave function* $h : I \to \mathbb{R}$ *such that*

$$\sup_{\lambda \in I} \inf_{u \in X} (\Phi(u) + \lambda \Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in I} (\Phi(u) + \lambda \Psi(u) + h(\lambda)).$$

Then, there exists an open interval $J \subseteq I$ *and a number* $\rho > 0$*, such that for each* $\lambda \in J$*, the equation* $\Phi'(u) + \lambda \Psi'(u) = 0$ *admits at least three distinct solutions in* X *whose norms are less than* ρ*.*

A refinement of the previous theorem is given by the following critical point result of Bonanno [20].

Theorem 6.1.5. (Bonanno [20]) *Let* $(X, \|\cdot\|)$ *be a separable and reflexive real Banach space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* $\Phi(u) \geq 0$ *for every* $u \in X$*. Furthermore, assume that there exist* $u_0, u_1 \in X$ *and* $\rho > 0$ *such that*

(1) $\Phi(u_0) = \Psi(u_0) = 0$,

(2) $< \ (u_1)$,

(3) $\sup\limits_{(u)<} \ (u) < \dfrac{(u_1)}{(u_1)}$.

Further, put

$$\bar{a} = \left(\frac{(u_1)}{(u_1)} \quad \sup_{(u)<} \ (u) \right)^{1} ;$$

where > 1, *and assume that the functional is sequentially weakly lower semi-continuous, satisfies the* (PS)*-condition, and*

(4) $\lim\limits_{kuk \ ! \ 1} (\ (u) \quad (u)) = +1$ *for every* $2 \ [0; \bar{a}]$.

Then, there exists an open interval $J \quad [0; \bar{a}]$ *and a number* > 0, *such that for each* $2 \ J$, *the equation* $^0(u) \quad ^0(u) = 0$ *admits at least three distinct solutions in* X *whose norms are less than* .

The next section demonstrates the application of the above variational principle by giving a multiplicity result for an elliptic PDE on a Randers space.

6.2 Multiple solutions for an elliptic PDE on Randers spaces

In this section let us consider a complete, n-dimensional Randers space $(M; F)$ with the Finsler structure $F : TM \ ! \ \mathrm{R}$,

$$F(x; v) = \sqrt{g_x(v; v)} + \ _x(v); \quad (x; v) \ 2 \ TM; \tag{6.1}$$

where g is a Riemannian metric and $_x$ is a 1-form on M. Recall that $j \ _x j_g = \sqrt{g_x(\ _x; \ _x)} < 1$, for every $x \ 2 \ M$, where g is the co-metric of g.

We consider the following parameter-dependent elliptic problem, where the leading term is given by the p-Finsler-Laplace operator $_{F;p}$, i.e.,

$$\begin{cases} \ _{F;p} u(x) = \ (x) h(u(x)); & x \ 2 \ M; \\ u \ 2 \ W^{1;p}_F(M); \end{cases} \tag{P }$$

where $n < p < 1$, is a positive parameter, $2 \ L^1(M) \setminus L^1(M)$, and $h : \mathrm{R} \ ! \ \mathrm{R}$ is a continuous function. For each $s \ 2 \ \mathrm{R}$, let $H(s) = \displaystyle\int_0^s h(t)\,\mathrm{d}t$. We assume the following properties:

(A_1) there exists $s_0 > 0$ such that $H(s) > 0; \ 8s \ 2 \ (0; s_0]$;

(A_2) there exist $C > 0$ and $1 < w < p$ such that $jh(s)j \quad C(1 + jsj^{w \ 1}); \ 8s \ 2 \ \mathrm{R}$;

(A_3) there exists $q > p$ such that

$$\limsup_{s\to 0}\frac{H(s)}{|s|^q} < \infty.$$

Then we can prove the following multiplicity result regarding problem (P_λ):

Theorem 6.2.1. *Let (M, F) be an n-dimensional Randers space endowed with the Finsler metric (6.1) such that $a := \sup_{x\in M}|\beta_x|_g < 1$ and g is a Riemannian metric, where (M, g) is a Hadamard manifold with sectional curvature bounded above by $-\kappa^2$, $\kappa > 0$. Suppose that G is a compact connected subgroup of $\mathrm{Isom}_F(M)$ such that $\mathrm{Fix}_M(G) = \{x_0\}$ for some $x_0 \in M$. Let $n < p < \infty$ and $\lambda > 0$ a parameter. If $h : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying (A_1)–(A_3) and $\alpha \in L^1(M)\cap L^\infty(M)$ is a nonzero, nonnegative function which depends on $d_F(x_0, \cdot)$ and satisfies*

$$\sup_{R>0}\ \operatorname*{essinf}_{d_F(x_0,x)\le R}\alpha(x) > 0,$$

then there exists an open interval $\Lambda \subset [0, \infty]$ and a number $\mu > 0$, such that for every $\lambda \in \Lambda$, problem (P_λ) admits at least three distinct solutions in $W^{1,p}_{F,G}(M)$ having $W^{1,p}_F(M)$-norms less than μ.

The proof is based on variational arguments, combining the compact embedding from Theorem 4.5.1 with the multiplicity result of Theorem 6.1.5. We divide the proof into several steps.

First, by Dirichlet's principle, one can associate the energy functional with problem (P_λ) for every $\lambda > 0$, namely

$$\mathcal{E}_\lambda : W^{1,p}_F(M) \to \mathbb{R}, \quad \mathcal{E}_\lambda(u) = \Phi_0(u) - \lambda\Psi_0(u),$$

where $\Phi_0, \Psi_0 : W^{1,p}_F(M) \to \mathbb{R}$,

$$\Phi_0(u) = \frac{1}{p}\int_M F^{*p}(x, Du(x))\,\mathrm{d}v_F(x) \quad\text{and}\quad \Psi_0(u) = \int_M \alpha(x)H(u(x))\,\mathrm{d}v_F(x).$$

The functional $\mathcal{E}_\lambda$ is well-defined and of class C^1 on $W^{1,p}_F(M)$. Furthermore, we have that

$$\mathcal{E}'_\lambda(u)(v) = \int_M Dv(\nabla_F u)(x)F^{*p-2}(x, Du(x))\,\mathrm{d}v_F(x) - \lambda\int_M \alpha(x)h(u(x))v(x)\,\mathrm{d}v_F(x),$$

for all $u, v \in W^{1,p}_F(M)$. Hence, it turns out that $\mathcal{E}'_\lambda(u) = 0$ if and only if $u \in W^{1,p}_F(M)$ is a weak solution of the problem (P_λ). Therefore, it suffices to study the critical points of the functional $\mathcal{E}_\lambda$.

Since (M, F) is a Randers space with $\sup_{x\in M}|\beta_x|_g < 1$, the reversibility constant r_F is finite, see (3.4). Then, by Theorem 2.7.1, we have that $W^{1,p}_F(M)$ forms a separable and reflexive Banach space with the associated symmetric norm $\|\cdot\|_{W^{1,p}_{F_s}(M)}$, while the norms $\|\cdot\|_{W^{1,p}_{F}(M)}$ and $\|\cdot\|_{W^{1,p}_{F_s}(M)}$ are equivalent, see Section 2.7.

In order to apply Theorem 6.1.5, we need some auxiliary results. First, we study the properties of the energy functional $\mathcal{E}_\lambda$.

Lemma 6.2.2. *Under the conditions of Theorem 6.2.1, the functional $\mathcal{E}_\lambda$ is coercive and bounded below.*

Proof. Using a McKean-type inequality on the Riemannian manifold (M,g) (see for instance Yin and He [127, Theorem 0.6]), we have that

$$\lambda^p_{1,g}(M) := \inf_{u\in W^{1,p}_g(M)\setminus\{0\}} \frac{\displaystyle\int_M |\nabla_g u|^p \,\mathrm{d}v_g}{\displaystyle\int_M |u|^p \,\mathrm{d}v_g} \geq \left(\frac{(n-1)}{p}\right)^p;$$

which yields that

$$\int_M |\nabla_g u|^p \,\mathrm{d}v_g \geq \frac{(n-1)^p\kappa^p}{p^p+(n-1)^p\kappa^p}\|u\|^p_{W^{1,p}_g(M)}, \quad \forall u\in W^{1,p}_g(M).$$

Using (4.22), (4.24), and denoting $c(n,a,p,\kappa) := \frac{(1-a^2)^{(n+1)/2}}{(1+a)^p}\frac{(n-1)^p\kappa^p}{p^p+(n-1)^p\kappa^p}$, we obtain that

$$\int_M F^{*p}(x,Du(x))\,\mathrm{d}v_F(x) \geq c(n,a,p,\kappa)\|u\|^p_{W^{1,p}_g(M)}, \quad \forall u\in W^{1,p}_g(M). \tag{6.2}$$

From (A_2), it follows that there exist $C, c_1 > 0$ and $1 < w < p$ such that

$$\mathcal{E}_\lambda(u) \geq \frac{c(n,a,p,\kappa)}{p}\|u\|^p_{W^{1,p}_g(M)} - \lambda C\|\alpha\|_{L^1(M)}\left(c_1\|u\|_{W^{1,p}_g(M)} + c_1^w\|u\|^w_{W^{1,p}_g(M)}\right);$$

for every $u\in W^{1,p}_g(M)$. Since $p > w$, the claim clearly follows. □

Lemma 6.2.3. *Under the conditions of Theorem 6.2.1, we have that $\mathcal{E}_\lambda$ is G-invariant, i.e., for every $\tau\in G$ and $u\in W^{1,p}_F(M)$ one has $\mathcal{E}_\lambda(\tau u) = \mathcal{E}_\lambda(u)$.*

Proof. We start with the G-invariance of the functional Φ_0. Since $(\tau u)(x) = u(\tau^{-1}x)$, by the chain rule, one has

$$\begin{aligned}
p\,\Phi_0(\tau u) &= \int_M F^{*p}(x, D(\tau u)(x))\,\mathrm{d}v_F(x)\\
&= \int_M F^{*p}(x, D(u(\tau^{-1}x)))\,\mathrm{d}v_F(x)\\
&= \int_M F^{*p}(x, Du(\tau^{-1}x)d\tau_x^{-1})\,\mathrm{d}v_F(x)\\
&= \int_M F^{*p}(\tau y, Du(y)d\tau_y^{-1})\,\mathrm{d}v_F(\tau y);
\end{aligned} \tag{6.3}$$

where in the last step we used a change of variable $^{-1}x = y$. First, since $\in G$, it follows that $\mathrm{d}V_F(\ y) = \mathrm{d}V_F(y)$. On the other hand, due to Deng and Hou [41], we have that

$$F(\ x; d\ _x(v)) = F(x; v);$$

for every $\in G; x \in M$ and $v \in T_xM$. Then, by the definition of the polar transform, we have that

$$\begin{aligned}
F\ (\ y; Du(y)d\ _y^{-1}) &= \sup_{w \in T\ _yM\setminus\{0\}} \frac{\big(Du(y)d\ _y^{-1}\big)(w)}{F(\ y; w)} \qquad (w := d\ _y(z);\ z \in T_yM) \\
&= \sup_{z \in T_yM\setminus\{0\}} \frac{Du(y)d\ _y^{-1}(d\ _y(z))}{F(\ y; d\ _y(z))} = \sup_{z \in T_yM\setminus\{0\}} \frac{Du(y)(z)}{F(y; z)} \\
&= F\ (y; Du(y)): \qquad (6.4)
\end{aligned}$$

Combining (6.3) and (6.4), we get the desired G-invariance of the functional $_0$.

Now, for the functional $_0$, we can write

$$\begin{aligned}
_0(\ u) &= \int_M (x)H((\ u)(x))\,\mathrm{d}v_F(x) \\
&= \int_M (x)H(u(\ ^{-1}x))\,\mathrm{d}v_F(x) \qquad (y := \ ^{-1}x) \\
&= \int_M (\ y)H(u(y))\,\mathrm{d}v_F(\ y):
\end{aligned}$$

Since $\mathrm{Fix}_M(G) = \{x_0\}$ and $\in L^1(M) \cap L^{\infty}(M)$ depends on $d_F(x_0; \ )$, it follows that for every $\in G$ and $u \in W_F^{1;p}(M)$, we have $_0(\ u) = \ _0(u)$, which concludes the proof. □

Having in our mind Theorems 4.5.1 and 6.1.5, we restrict the energy functional E to the space $W_{F;G}^{1;p}(M)$. Since G is a compact connected subgroup of $\mathrm{Isom}_F(M)$, $W_{F;G}^{1;p}(M)$ forms a closed linear subspace of $W_F^{1;p}(M)$, see Kobayashi and Ôtani [68]. Therefore, $W_{F;G}^{1;p}(M)$ turns out to be a separable and reflexive Banach space as well. For simplicity, in the following we use the notations

$$E\ := E\ |_{W_{F;G}^{1;p}(M)}; \qquad := \ _0|_{W_{F;G}^{1;p}(M)} \quad \text{and} \quad := \ _0|_{W_{F;G}^{1;p}(M)}:$$

Since E is G-invariant, the principle of symmetric criticality of Palais (see Kobayashi and Ôtani [68] and Kristály, Rădulescu, and Varga [79, Theorem 1.50]) implies that the critical points of E are also critical points of the original functional E. Therefore, it is enough to find critical points of E.

Lemma 6.2.4. *Under the assumptions of Theorem 6.2.1, E satisfies the Palais–Smale condition on $W_{F;G}^{1;p}(M)$.*

Proof. Let $(u_k)_k$ be a sequence in $W^{1,p}_{F,G}(M)$ such that the sequence $(E(u_k))_k$ is bounded and $\|E'(u_k)\| \to 0$ when $k \to \infty$. Since E is coercive, the sequence $(u_k)_k$ is bounded in $W^{1,p}_{F,G}(M)$. Therefore, up to a subsequence, $u_k \rightharpoonup u$ in $W^{1,p}_{F,G}(M)$ for some $u \in W^{1,p}_{F,G}(M)$. Hence, due to Theorem 4.5.1 and Theorem 4.3.3, it follows that $u_k \to u$ strongly in $L^{1}(M)$. In particular, we have that

$$E'(u)(u - u_k) \to 0 \quad \text{and} \quad E'(u_k)(u - u_k) \to 0 \quad \text{as} \quad k \to \infty. \tag{6.5}$$

On the one hand, it is easy to verify that

$$\int_M (Du(x) - Du_k(x))\big(\nabla_F u(x) F^{p-2}(x; Du(x)) - \nabla_F u_k(x) F^{p-2}(x; Du_k(x))\big)\, dv_F(x)$$
$$= E'(u)(u - u_k) - E'(u_k)(u - u_k) + \int_M (x)[h(u_k) - h(u)](u_k(x) - u(x))\, dv_F(x).$$

On the other hand, we have

$$\left|\int_M (x)[h(u_k) - h(u)](u_k(x) - u(x))\, dv_F(x)\right|$$
$$\leq 2\| \|_{L^1(M)} \max\{|h(s)| : |s| \leq \|u\|_{L^{\infty}(M)} + 1\}\|u_k - u\|_{L^{\infty}(M)}. \tag{6.6}$$

The mean value theorem implies that for all $x \in M$,

$$(Du(x) - Du_k(x))(\nabla_F u(x) F^{p-2}(x; Du(x)) - \nabla_F u_k(x) F^{p-2}(x; Du_k(x)))$$
$$\geq l_F F^{p}(x; Du(x) - Du_k(x)),$$

where l_F is the uniformity constant associated to F (see (3.5)). Since (M, F) is a Randers space with $\sup_{x \in M} |\ _x|_g < 1$, it follows that $l_F > 0$, therefore $u_k \to u$ in $W^{1,p}_{F,G}(M)$, which proves the claim. □

Lemma 6.2.5. *Under the assumptions of Theorem 6.2.1, the functional E is sequentially weakly lower semi-continuous.*

Proof. Since is a norm-type functional, it follows that is sequentially weakly lower semi-continuous. Therefore, it suffices to prove that is sequentially weakly continuous. To this end, consider a sequence $(u_k)_k$ in $W^{1,p}_{F,G}(M)$ which converges weakly to $u \in W^{1,p}_{F,G}(M)$, and suppose that

$$\lim_{k \to \infty} (u_k) \neq (u).$$

Therefore, due to Theorem 4.5.1 and Theorem 4.3.3, there exist $\varepsilon > 0$ and a subsequence of $(u_k)_k$, denoted by $(u_m)_m$, such that $u_m \to u$ in $L^{1}(M)$ and

$$0 < \varepsilon \leq | (u_m) - (u)|, \quad \text{for every } m \in \mathbb{N}.$$

Thus, by the mean value theorem (see also (6.6)), for each $m \in \mathbb{N}$ there exists $\theta_m \in (0,1)$ such that

$$
\begin{aligned}
0 < \varepsilon &\leq \left| \Phi'(u+\theta_m(u_m-u))(u_m-u)\right| \\
&\leq \int_M \alpha(x)\left|h(u(x)+\theta_m(u_m(x)-u(x)))\right| |u_m(x)-u(x)| \,\mathrm{d}v_F(x) \\
&\leq \|\alpha\|_{L^1(M)} \max\{|h(s)| : |s| \leq \|u\|_{L^\infty(M)}+1\} \|u_m-u\|_{L^\infty(M)}.
\end{aligned}
$$

Note that the last term tends to 0, which yields a contradiction. □

Now, we are in the position to prove Theorem 6.2.1.

Proof of Theorem 6.2.1. Let $s_0 > 0$ be given by condition (A_1). Recall that

$$\sup_{R>0} \operatorname*{essinf}_{d_F(x_0,x)\leq R} \alpha(x) > 0,$$

thus we may choose an $R > 0$ such that $\alpha_R := \operatorname*{essinf}_{d_F(x_0,x)\leq R} \alpha(x) > 0.$ Then, for a fixed $r < R\dfrac{1-a}{1+a}$, where $0 \leq a = \sup_{x\in M} |\beta_x|_g < 1$, let us define the function $u_{s_0,R,r} : M \to \mathbb{R}$,

$$
u_{s_0,R,r} = \begin{cases} 0, & x \in M \setminus B_F(x_0,R); \\ \frac{s_0}{R-r}(R - d_F(x_0,x)), & x \in B_F(x_0,R)\setminus B_F(x_0,r); \\ s_0, & x \in B_F(x_0,r). \end{cases}
$$

Since $a < 1$, recall that the reversibility constant of (M,F) is $r_F < \infty$, see (3.4). Therefore, by the eikonal identity (2.3), we have that $\frac{1}{r_F} \leq F(x, -Dd_F(x_0,x)) \leq r_F.$ Hence, it follows that

$$
\begin{aligned}
\left(\frac{s_0}{R-r}\right)^p \frac{1}{r_F^p}\left(\mathrm{Vol}_F(B_F(x_0,R)) - \mathrm{Vol}_F(B_F(x_0,r))\right) &\leq \int_M F^{*p}(x, Du_{s_0,R,r}(x))\,\mathrm{d}v_F(x) \\
&\leq \left(\frac{s_0}{R-r}\right)^p r_F^p \mathrm{Vol}_F(B_F(x_0,R)).
\end{aligned}
$$

Since $0 \leq u_{s_0,R,r}(x) \leq s_0$, for all $x \in M$, using hypothesis (A_1) yields that

$$
\begin{aligned}
\Psi(u_{s_0,R,r}) = \int_M \alpha(x)H(u_{s_0,R,r}(x))\,\mathrm{d}v_F(x) &= \int_{B_F(x_0,R)} \alpha(x)H(u_{s_0,R,r}(x))\,\mathrm{d}v_F(x) \\
&\geq \int_{B_F(x_0,r)} \alpha(x)H(u_{s_0,R,r}(x))\,\mathrm{d}v_F(x) = H(s_0)\alpha_R \mathrm{Vol}_F(B_F(x_0,r)) > 0.
\end{aligned}
$$

On the one hand, by (A_3), we may fix $s_1 \in (0,1]$ and $C_1 > 0$ such that

$$H(s) \leq C_1|s|^q, \quad \text{whenever } |s| < s_1.$$

On the other hand, by (A_2), we have that

$$|H(s)| \le C(1+|s|^{w-1})|s| \le C\frac{1+s_1^{w-1}}{s_1^{q-1}}|s|^q; \quad \text{for all } |s| \ge s_1.$$

Choosing $C_2 = \max\left\{C_1, C\frac{1+s_1^{w-1}}{s_1^{q-1}}\right\}$, we obtain that

$$H(s) \le C_2|s|^q, \quad \forall s \in \mathbb{R}.$$

Therefore,

$$\Psi(u) = \int_M \beta(x)H(u(x))\,\mathrm{d}v_F(x) \le C_2\|\beta\|_{L^1(M)}c_1^q\|u\|^q_{W_g^{1,p}(M)} \tag{6.7}$$

for some $c_1 > 0$. Now, we claim that

$$\limsup_{\lambda \to 0} \frac{\sup\left\{\Psi(u) : \int_M F^{*p}(x, Du(x))\,\mathrm{d}v_F(x) < p\lambda\right\}}{\lambda} \le 0. \tag{6.8}$$

To prove the previous assertion, first observe that by (6.2), we have that

$$\frac{\sup\left\{\Psi(u) : \int_M F^{*p}(x, Du(x))\,\mathrm{d}v_F(x) < p\lambda\right\}}{\lambda} \le \frac{\sup\left\{\Psi(u) : c(n,a,p,\epsilon)\|u\|^p_{W_g^{1,p}(M)} < p\lambda\right\}}{\lambda}.$$

By applying (6.7), it follows that

$$\frac{\sup\left\{\Psi(u) : c(n,a,p,\epsilon)\|u\|^p_{W_g^{1,p}(M)} < p\lambda\right\}}{\lambda} < \frac{C_2\|\beta\|_{L^1(M)}c_1^q\left(\frac{p\lambda}{c(n,a,p,\epsilon)}\right)^{\frac{q}{p}}}{\lambda} \to 0$$

when $\lambda \to 0$, where we used the fact that $q > p$, thus relation (6.8) holds.

Then, we may choose $\lambda_0 > 0$ such that

$$\lambda_0 < \frac{1}{p}c(n,a,p,\epsilon)\|u_{s_0,R,r}\|^p_{W_g^{1,p}(M)} \le \frac{1}{p}\int_M F^{*p}(x, Du_{s_0,R,r}(x))\,\mathrm{d}v_F(x),$$

and

$$\frac{\sup\left\{\Psi(u) : \int_M F^{*p}(x, Du(x))\,\mathrm{d}v_F(x) < p\lambda_0\right\}}{\lambda_0} < \frac{\Psi(u_{s_0,R,r})}{\Phi(u_{s_0,R,r})}.$$

Now we shall apply Theorem 6.1.5 to the functionals $\Phi, \Psi : W^{1,p}_{F,G}(M) \to \mathbb{R}$. Let $u_0 = 0$, $u_1 = u_{s_0,R,r}$ and $\rho = \lambda_0$. First, observe that the conditions (1), (2) and (3) of Theorem

6.1.5 are satisfied. Furthermore, define

$$\bar{a} = (1 + \ _0)\left(\frac{(u_{s_0;R;r})}{(u_{s_0;R;r})} \quad \frac{\sup f \quad (u) : \quad (u) < \ _0 g}{_0} \right)^{1} :$$

Taking into account Lemmas 6.2.2, 6.2.4 and 6.2.5, we obtain that all the assumptions of Theorem 6.1.5 are verified. Thus, there exists an open interval $[0; \bar{a}]$ and a number > 0, such that for each 2, the equation $E^0(u) = \ ^0(u) \quad ^0(u) = 0$ admits at least three distinct solutions in $W^{1;p}_{F;G}(M)$ having $W^{1;p}_F(M)$-norms less than . By the principle of symmetric criticality, this concludes the proof. □

Chapter 7

Summary

In this chapter we formulate the theses which describe the main contributions of the present dissertation.

1 Three isometrical models of Finsler manifolds

Main result: A new isometric model of the Finsler-Poincaré disk.

1.1 Thesis

The Finsler-Poincaré upper half plane introduced by [3] represents the Finslerian Randers-type generalization of the Riemannian hyperbolic upper half plane.

1.2. Thesis

The Finsler-Poincaré disk model, the 2-dimensional Funk model and the Finsler-Poincaré upper half plane are isometrically equivalent, see [3]. This isometry result represents a natural extension of the isometrical equivalence of the Riemannian hyperbolic model spaces, more precisely, the Riemannian Poincaré disk, the Beltrami-Klein disk and the Poincaré upper half plane model.

1.3. Thesis

In the case of the Finsler-Poincaré disk, the 2-dimensional Funk model and the Finsler-Poincaré upper half plane, the first eigenvalue associated to the Finsler-Laplace operator turns out to be zero, see [3]. This gapless character of the fundamental frequency is in sharp contrast with the Riemannian spectral gap property proved by McKean [90].

2 Sobolev-type inequalities without singular terms

Main results: Compact embeddings on noncompact Riemannian manifolds and Randers spaces.

2.1. Thesis

Given a noncompact complete Riemannian manifold (M, g), we introduce an expansion condition concerning the action of a compact connected group G which represents a subgroup of the isometry group of (M, g), see [1]. Then, in the case when (M, g) is either a Cartan-Hadamard manifold or a manifold with bounded geometry, the previous expansion condition characterizes the coerciveness of G in the sense of Skrzypczak and Tintarev [111].

2.2. Thesis

If (M, g) is either a Cartan-Hadamard manifold or a Riemannian manifold with bounded geometry and G is a compact connected subgroup of the isometry group of (M, g), then compact Berestycki-Lions-type Sobolev embeddings hold for the full range of admissible parameters (i.e., Sobolev, Moser-Trudinger and Morrey case) whenever the above expansion condition is satisfied, see [1].

2.3. Thesis

Randers spaces with finite reversibility constant inherit similar compact embedding properties whenever the underlying Riemannian manifold is either Cartan-Hadamard manifold or has bounded geometry, and the group of isometries G satisfies a similar expansion condition, see [1]. Moreover, the finiteness of the reversibility constant turns out to be an indispensable condition for the validity of the continuous Sobolev embeddings. This property is demonstrated by the counterexample given on the n-dimensional Funk model, where the continuous Sobolev embeddings fail to hold for each admissible parameter pair, rendering the compact embeddings unattainable, see [1].

3 Sobolev-type inequalities with singular terms

Main results: Hardy-type inequalities on Finsler manifolds.

3.1. Thesis

If (M, F) is a forward complete Finsler manifold, then the superharmonicity of a certain nonnegative weight function provides a sufficient condition for several weighted Hardy-type inequalities on (M, F), which form the Finslerian counterparts of the results due to

D'Ambrosio and Dipierro [38]. In this setting, we obtain among others a weighted Hardy inequality, a weighted Gagliardo-Nirenberg inequality and a weighted Heisenberg-Pauli-Weyl uncertainty principle, see [4].

3.2. Thesis

In the case of a Finsler-Hadamard manifold $(M;F)$ with finite reversibility constant and vanishing mean covariation, [4] establishes the generalization of the classical Euclidean Hardy inequality, which features the Finslerian distance function from an arbitrarily fixed point $x_0 \in M$ and the reversibility constant of $(M;F)$. This construction represents the quantitative version of the inequality given by Zhao [132, Theorem 1.2] and the generalization of the result due to Farkas, Kristály, and Varga [50, Proposition 4.1].

3.3. Thesis

Given a complete Finsler manifold with nonzero uniformity constant (thus finite reversibility constant), in [2] we prove a bipolar Hardy inequality featuring two singularities, which represents the Finslerian generalization of the Riemannian multipolar inequality proved by Faraci, Farkas, and Kristály [48] and the Euclidean version due to Cazacu and Zuazua [32]; the constants appearing in the bipolar Hardy inequality are dependent on the reversibility constant and uniformity constant of $(M;F)$.

4 Application to partial differential equations

Main result: Multiplicity result concerning an elliptic problem defined on a Randers space with finite reversibility constant.

4.1. Thesis

The obtained Sobolev-type inequalities provide means for studying elliptic PDEs via variational methods. In particular, we prove a multiplicity result concerning a quasilinear problem defined on a Randers space with finite reversibility constant, where the leading term is given by the p-Finsler-Laplace operator. More precisely, under certain growth conditions, the studied parameter-dependent quasilinear elliptic PDE has three distinct solutions for parameters small enough, see [1]. The stability of the solutions is ensured by a Ricceri-type critical point theorem due to Bonanno [20].

Bibliography

[5] R. A. Adams and J. J. F. Fournier. “Sobolev spaces”. *Pure and Applied Mathematics (Amsterdam)*. Vol. 140. Elsevier/Academic Press, 2003.

[6] A. Alvino et al. “Convex symmetrization and applications”. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 14.2 (1997), pp. 275–293.

[7] A. Ambrosetti and P. Rabinowitz. “Dual variational methods in critical point theory and applications”. *J. Funct. Anal.* 14 (1973), pp. 349–381.

[8] A. Andai. *Information geometry in quantum mechanics*. PhD thesis. Department of Analysis, Budapest University of Technology and Economics; in Hungarian, 2003.

[9] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto. *The theory of sprays and Finsler spaces with applications in physics and biology*. Fundamental Theories of Physics 58. Springer, 1993.

[10] T. Aubin. “Espaces de Sobolev sur les variétés Riemanniennes”. *Bull. Sci. Math.* 100 (1976), pp. 149–173.

[11] T. Aubin. “Nonlinear analysis on manifolds. Monge-Ampère equations”. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Vol. 252. Springer-Verlag, New York, 1982.

[12] T. Aubin. “Problèmes isopérimétriques et espaces de Sobolev”. *J. Di erential Geom.* 11.4 (1976), pp. 573–598.

[13] A.A. Balinsky, W.D. Evans, and R.T. Lewis. *The Analysis and Geometry of Hardy’s Inequality*. Universitext, Springer-Verlag, 2015.

[14] Z. Balogh and A. Kristály. “Sharp isoperimetric and Sobolev inequalities in spaces with nonnegative Ricci curvature”. *Math. Annalen. Accepted* (2022). doi: 10.1007/s00208-022-02380-1.

[15] D. Bao, S.-S. Chern, and Z. Shen. “An Introduction to Riemann-Finsler Geometry”. *Graduate Texts in Mathematics: vol. 200*. Springer-Verlag, 2000.

[16] D. Bao and C. Robles. “Ricci and flag curvatures in Finsler geometry”. *Riemann-Finsler Geometry, MSRI Publications* 50 (2004).

[17] D. Bao, C. Robles, and Z. Shen. “Zermelo navigation on Riemannian manifolds”. *J. Di erential Geom.* 66.3 (2004), pp. 377–435.

[18] H. Berestycki and P.-L. Lions. “Existence of a ground state in nonlinear equations of the Klein-Gordon type”. *Variational inequalities and complementarity problems (Proc. Internat. School, Erice, 1978)* (1980), pp. 35–51.

[19] R. L. Bishop and R. J. Crittenden. “Geometry of manifolds”. *Pure and Applied Mathematics*. Vol. XV. Academic Press, New York-London, 1964.

[20] G. Bonanno. “Some remarks on a three critical points theorem”. *Nonlinear Anal.* 54.4 (2003), pp. 651–665.

[21] R. Bosi, J. Dolbeault, and M. J. Esteban. “Estimates for the optimal constants in multipolar Hardy inequalities for Schrödinger and Dirac operators”. *Commun. Pure and Appl. Math.* 7 (2008), pp. 533–562.

[22] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, New York, 2011.

[23] H. Brezis and M. Marcus. “Hardy’s inequalities revisited”. *Ann. Scuola Norm. Sup. Cl. Sci.* 25.4 (1997), pp. 217–237.

[24] H. Brezis and J.L. Vázquez. “Blow-up solutions for some nonlinear elliptic problems”. *Rev. Mat. Univ. Complutense Madrid* 10.2 (1997), pp. 443–469.

[25] L. Caffarelli, R. Kohn, and L. Nirenberg. “First order interpolation inequalities with weight”. *Compos. Math.* 53 (1984), pp. 259–275.

[26] J. W. Cannon et al. “Hyperbolic Geometry”. *MSRI Publications*. Vol. 31. 1997, pp. 59–115.

[27] M. Cantor. “Sobolev inequalities for Riemannian bundles”. 80 (1974), pp. 239–243.

[28] E. Caponio, A. V. Germinario, and M. Sánchez. “Convex regions of stationary spacetimes and Randers spaces. Applications to lensing and asymptotic flatness”. *The Journal of Geometric Analysis* 26 (2016), pp. 791–836.

[29] E. Caponio, M.A. Javaloyes, and M. Sánchez. “Wind Finslerian structures: from Zermelo’s navigation to the causality of spacetimes”. *Memoirs Amer. Math. Soc. To appear* (2021). arXiv: 1407.5494.

[30] M. P. do Carmo. “Riemannian geometry”. *Mathematics: Theory & Applications*. Translated by F. Flaherty. Birkhäuser Basel, 1992.

[31] G. Carron. “Inégalités de Hardy sur les variétés Riemanniennes non-compactes”. *J. Math. Pures Appl.* 76.10 (1997), pp. 883–891.

[32] C. Cazacu and E. Zuazua. “Improved multipolar Hardy inequalities”. *Studies in phase space analysis with applications to PDEs, Progr. Nonlinear Differential Equations Appl.* 84 (2013), pp. 35–52.

[33] I. Chavel. “Eigenvalues in Riemannian geometry”. *Pure and Applied Mathematics 115.* Academic Press, 1984.

[34] X. Cheng and Z. Shen. *Finsler geometry. An approach via Randers spaces*. Springer-Verlag, 2012.

[35] Y. Cho and T. Ozawa. “Sobolev inequalities with symmetry”. *Commun. Contemp. Math.* 11.3 (2009), pp. 355–365.

[36] S. Cobzaş. *Functional Analysis in Asymmetric Normed Spaces*. Birkhäuser/Springer Basel, Basel, 2013.

[37] C. Croke. “A sharp four-dimensional isoperimetric inequality”. *Comment. Math. Helv.* 59.2 (1984), pp. 187–192.

[38] L. D’Ambrosio and S. Dipierro. “Hardy inequalities on Riemannian manifolds and applications”. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 31.3 (2014), pp. 449–475.

[39] H. R. Dehkordi. “Mathematical modeling of wildfire propagation in an agricultural land”. *Proceeding Series of the Brazilian Society of Computational and Applied Mathematics* 8.1 (2021). doi: 10.5540/03.2021.008.01.0431.

[40] S. Deng. *Homogeneous Finsler spaces*. Springer, New York, 2012.

[41] S. Deng and Z. Hou. “The group of isometries of a Finsler space”. *Pacific J. Math.* 207.1 (2002), pp. 149–155.

[42] O. Druet and E. Hebey. “The AB program in geometric analysis: sharp Sobolev inequalities and related problems”. *Memoirs of the American Mathematical Society*. Vol. 160. 761. American Mathematical Society, 2002.

[43] O. Druet, E. Hebey, and M. Vaugon. “Optimal Nash’s inequalities on Riemannian manifolds: the influence of geometry”. 14 (1999), pp. 735–779.

[44] Y. Ebihara and T. P. Schonbek. “On the (non)compactness of the radial Sobolev spaces”. *Hiroshima Math. J.* 16.3 (1986), pp. 665–669.

[45] D. Eglo. “Uniform Finsler Hadamard manifolds”. *Ann. Inst. H. Poincaré Phys. Théor.* 66.3 (1997), pp. 323–357.

[46] I. Ekeland. “On the variational principle”. *J. Math. Anal. Appl.* 47 (1974), pp. 324–353.

[47] L. C. Evans. “Partial Dierential Equations. Second edition”. *Graduate Studies in Mathematics*. Vol. 19. American Mathematical Society, 2010.

[48] F. Faraci, C. Farkas, and A. Kristály. “Multipolar Hardy inequalities on Riemannian manifolds”. *ESAIM: Control Optim. and Calc. of Variations* 24.2 (2018), pp. 551–567.

[49] C. Farkas, J. Fodor, and A. Kristály. “Anisotropic elliptic problems involving sublinear terms”. *2015 IEEE 10th Jubilee International Symposium on Applied Computational Intelligence and Informatics* (2015), pp. 141–146.

[50] C. Farkas, A. Kristály, and C. Varga. “Singular Poisson equations on Finsler-Hadamard manifolds”. *Calc. Var. and PDE* 54.2 (2015), pp. 1219–1241.

[51] V. Felli, E. M. Marchini, and S. Terracini. “On Schrödinger operators with multipolar inverse-square potentials”. *J. Funct. Anal.* 250.2 (2007), pp. 265–316.

[52] Vincenzo Ferone and Bernd Kawohl. “Remarks on a Finsler-Laplacian”. *Proceedings of the American Mathematical Society* 250.2 (2009), pp. 247–253.

[53] P. Finsler. *Über Kurven und Flächen in allgemeinen Räumen*. Dissertation, Göttingen, 1918. (Basel, 1951).

[54] M. Gaczkowski, P. Górka, and D. J. Pons. “Sobolev spaces with variable exponents on complete manifolds”. *J. Funct. Anal.* 270.4 (2016), pp. 1379–1415.

[55] E. Gagliardo. “Proprietà di alcune classi di funzioni in più variabili”. *Ricerche Mat.* 7 (1958), pp. 102–137.

[56] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Springer-Verlag, Berlin, third edition, 2004.

[57] N. Ghoussoub and A. Moradifam. “Bessel pairs and optimal Hardy and Hardy-Rellich inequalities”. *Math. Ann.* 349.1 (2011), pp. 1–57.

[58] G. W. Gibbons and C. M. Warnick. “The geometry of sound rays in a wind”. *Contemp. Phys.* 52 (2011), pp. 197–209.

[59] P. Górka. “Looking for compactness in Sobolev spaces on noncompact metric spaces”. *Ann. Acad. Sci. Fenn. Math.* 43.1 (2018), pp. 531–540.

[60] Q. Guo, J. Han, and P. Niu. “Existence and multiplicity of solutions for critical elliptic equations with multi-polar potentials in symmetric domains”. *Nonlinear Anal.* 75.15 (2012), pp. 5765–5786.

[61] G.H. Hardy. “Notes on some points in the integral calculus, LX. An inequality between integrals”. *Messenger Math.* 54 (1925), pp. 150–156.

[62] G.H. Hardy, J.E. Littlewood, and G. Pólya. *Inequalities, 2nd ed.* Cambridge University Press, Cambridge, 1952.

[63] E. Hebey. “Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities”. *Courant Lecture Notes in Mathematics*. Vol. 5. American Mathematical Society, 2000.

[64] E. Hebey and M. Vaugon. “Sobolev spaces in the presence of symmetries”. *J. Math. Pures Appl. (9)* 76.10 (1997), pp. 859–881.

[65] H. Ishikawa. “Note on Finslerian relativity”. *J. Math. Phys.* 22 (1981), pp. 995–1004.

[66] B. Kleiner. “An isoperimetric comparison theorem”. *Invent. Math.* 108.1 (1992), pp. 37–47.

[67] W. P. A. Klingenberg. *Riemannian geometry*. Vol. 1. Walter de Gruyter & Co., Berlin, second edition, 1995.

[68] J. Kobayashi and M. Ôtani. "The principle of symmetric criticality for non-differentiable mappings". *J. Funct. Anal.* 214.2 (2004), pp. 428–449.

[69] I. Kombe and M. Özaydin. "Hardy-Poincaré, Rellich and uncertainty principle inequalities on Riemannian manifolds". *Trans. Amer. Math. Soc.* 365.10 (2013), pp. 5035–5050.

[70] I. Kombe and M. Özaydin. "Improved Hardy and Rellich inequalities on Riemannian manifolds". *Trans. Amer. Math. Soc.* 361.12 (2009), pp. 6191–6203.

[71] P. Kopacz. "On generalization of Zermelo navigation problem on Riemannian manifolds". *International Journal of Geometric Methods in Modern Physics* 16.4 (2019), pp. 1950058.

[72] A. Kristály. "A Sharp Sobolev Interpolation Inequality on Finsler Manifolds". *J. Geom. Anal.* 25 (2014), pp. 2226–2240.

[73] A. Kristály. "Infinitely many solutions for a differential inclusion problem in R^N". *J. Di erential Equations* 220.2 (2006), pp. 511–530.

[74] A. Kristály. "Nash-type equilibria on Riemannian manifolds: a variational approach". *J. Math. Pures Appl.* 101.5 (2014), pp. 660–688.

[75] A. Kristály. "New features of the first eigenvalue on negatively curved spaces". *Adv. Calc. Var.* 15.3 (2022), pp. 475–495.

[76] A. Kristály. "New geometric aspects of Moser-Trudinger inequalities on Riemannian manifolds: the non-compact case". *J. Funct. Anal.* 276.8 (2019), pp. 2359–2396.

[77] A. Kristály. "Sharp Morrey-Sobolev inequalities on complete Riemannian manifolds". *Potential Anal.* 42.1 (2015), pp. 141–154.

[78] A. Kristály, Á. Mester, and I. I. Mezei. "Sharp Morrey-Sobolev inequalities and eigenvalue problems on Riemannian-Finsler manifolds with nonnegative Ricci curvature". *Communications in Contemp. Math. Accepted* (2022). DOI: 10.1142/S0219199722500638.

[79] A. Kristály, V. D. Rădulescu, and C. G. Varga. "Variational principles in mathematical physics, geometry, and economics". *Encyclopedia of Mathematics and its Applications*. Vol. 136. Cambridge University Press, 2010.

[80] A. Kristály and D. Repovš. "Quantitative Rellich inequalities on Finsler-Hadamard manifolds". *Commun. Contemp. Math.* 18.6 (2016), p. 1650020.

[81] A. Kristály and I. Rudas. "Elliptic problems on the ball endowed with Funk-type metrics". *Nonlinear Anal.* 119 (2015), pp. 199–208.

[82] A. Kufner, L. Maligranda, and L.-E. Persson. *The Hardy Inequality: About Its History and Some Related Results*. Vydavatelský servis, Pilsen, 2007.

[83] E. H. Lieb and M. Loss. “Analysis”. *Graduate Studies in Mathematics*. Vol. 14. American Mathematical Society, Providence, 2001.

[84] P.-L. Lions. “Symétrie et compacité dans les espaces de Sobolev”. *J. Funct. Anal.* 49.3 (1982), pp. 315–334.

[85] J. Lott and C. Villani. “Ricci curvature for metric measure spaces via optimal transport”. *Annals of Mathematics* 169.3 (2009), pp. 903–991.

[86] B. Loustau. *Hyperbolic geometry. To appear*. 2021. arXiv: 2003.11180.

[87] M. Matsumoto. “A slope of a mountain is a Finsler surface with respect to a time measure”. *J. Math. Kyoto Univ.* 29 (1989), pp. 17–25.

[88] V. G. Maz’ya. “Classes of domains and embedding theorems for function spaces”. *Dokl. Akad. Nauk. SSSR* 133 (1960), pp. 527–530. English transl.: Sov. Math. Dokl. 1, pp. 882–885.

[89] V. G. Maz’ya. “Sobolev Spaces with Applications to Elliptic Partial Di□erential Equations. Second, revised and augmented edition.” *Grundlehren der mathematischen Wissenschaften*. Vol. 342. Springer, 2011.

[90] H. P. McKean. “An upper bound to the spectrum of on a manifold of negative curvature”. *J. Di erential Geom.* 4.3 (1970), pp. 359–366.

[91] M. Moakher and M. Zéraï. “The Riemannian Geometry of the Space of Positive-Definite Matrices and Its Application to the Regularization of Positive-Definite Matrix-Valued Data”. *J. Math. Imaging Vis.* 40 (2010), pp. 171–187.

[92] C. B. Morrey. “Functions of several variables and absolute continuity, II”. *Duke J. Math.* 6 (1940), pp. 187–215.

[93] M. Muratori and N. Soave. “Some rigidity results for Sobolev inequalities and related PDEs on Cartan-Hadamard manifolds”. *Ann. Scuola Norm. Sup. Cl. Sci.* (2021), pp. 30. doi: 10.2422/2036-2145.202105_071.

[94] S. B. Myers and N. Steenrod. “The group of isometries of a Riemannian manifold”. *Ann. Math.* 40 (1939), pp. 400–416.

[95] L. Nirenberg. “On elliptic partial di□erential equations”. *Ann. Scuola Norm. Sup. Pisa (3)* 13 (1959), pp. 115–162.

[96] S. Ohta. “Finsler interpolation inequalities”. *Calc. Var. and PDE* 36 (2009), pp. 211–249.

[97] S. Ohta. “Some functional inequalities on non-reversible Finsler manifolds”. *Proc. Indian Acad. Sci. (Math. Sci.)* 127.5 (2017), pp. 833–855. Correction in: 131 (2021), Article No. 23, 2pp.

[98] S. Ohta. "Uniform convexity and smoothness, and their applications in Finsler geometry". *Math. Ann.* 343.3 (2009), pp. 669–699.

[99] S. Ohta and K.-T. Sturm. "Heat flow on Finsler manifolds". *Comm. Pure Appl. Math.* 62.10 (2009), pp. 1386–1433.

[100] P. Pucci and J. Serrin. "A mountain pass theorem". *J. Di erential Equations* 60 (1985), pp. 142–149.

[101] H.-B. Rademacher. "A sphere theorem for non-reversible Finsler metrics". *Math. Ann.* 328.3 (2004), pp. 373–387.

[102] G. Randers. "On an asymmetrical metric in the four-space of general relativity". *Phys. Rev.* 59.2 (1941), pp. 195–199.

[103] F. Rellich. "Halbeschränkte Di erentialoperatoren höherer Ordnung". *Proceedings of the International Congress of Mathematicians* III. (1954), pp. 243–250.

[104] B. Ricceri. "Existence of three solutions for a class of elliptic eigenvalue problems". *Math. Comput. Modelling* 32.11–13 (2000), pp. 1485–1494.

[105] B. Ricceri. "On a three critical points theorem". *Arch. Math. (Basel)* 75.3 (2000), pp. 220–226.

[106] W. Rudin. "Principles of Mathematical Analysis". *International Series in Pure and Applied Mathematics*. McGraw Hill, 3rd edition, 2013.

[107] S. F. Rutz and P. J. McCarthy. "A Finsler perturbation of the Poincaré metric". *General Relativity and Gravitation* 25 (1993), pp. 179–187.

[108] Z. Shen. "Finsler metrics with $K = 0$ and $S = 0$". *Canadian J. Math.* 55 (2003), pp. 112–132.

[109] Z. Shen. *Lectures on Finsler geometry*. World Scientific, 2001.

[110] Z. Shen. "Volume comparison and its applications in Riemann-Finsler geometry". *Advances in Mathematics* 128 (1997), pp. 306–328.

[111] L. Skrzypczak and C. Tintarev. "A geometric criterion for compactness of invariant subspaces". *Arch. Math. (Basel)* 101.3 (2013), pp. 259–268.

[112] L. Skrzypczak and C. Tintarev. "On compact subsets of Sobolev spaces on manifolds". *Trans. Amer. Math. Soc.* 374 (2021), pp. 3761–3777.

[113] S. L. Sobolev. "Sur un théorème d'analyse fonctionnelle". *Mat. Sb. (N.S)* 46 (1938), pp. 471–496.

[114] S. Stahl. *The Poincaré half-plane; A gateway to modern geometry*. Jones and Bartlett, 1993.

[115] W. A. Strauss. "Existence of solitary waves in higher dimensions". *Comm. Math. Phys.* 55 (1977), pp. 149–162.

[116] K.-T. Sturm. “On the geometry of metric measure spaces. I”. *Acta Mathematica* 196.1 (2006), pp. 65–131.

[117] K.-T. Sturm. “On the geometry of metric measure spaces. II”. *Acta Mathematica* 196.1 (2006), pp. 133–177.

[118] G. Talenti. “Best constant in Sobolev inequality”. *Annali di Matematica Pura ed Applicata* 110 (1976), pp. 353–372.

[119] C. Tintarev. *Concentration Compactness: Functional-Analytic Theory of Concentration Phenomena*. De Gruyter, 2020.

[120] N. T. Varopoulos. “Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique”. *Bull. Sci. Math.* 113.3 (1989), pp. 253–277.

[121] C. Villani. “Optimal transport. Old and new”. *Grundlehren der Mathematischen Wissenschaften*. Vol. 338. 2009.

[122] A. Weil. “Sur les surfaces a courbure négative”. *C. R. Acad. Sci. Paris* 182.2 (1926), pp. 1069–1071.

[123] M. Willem. “Minimax theorems”. *Progress in Nonlinear Differential Equations and their Applications*. Vol. 24. Birkhäuser Boston, 1996.

[124] B.Y. Wu and Y.L. Xin. “Comparison theorems in Finsler geometry and their applications”. *Math. Ann.* 337.1 (2007), pp. 177–196.

[125] C. Xia. “Hardy and Rellich type inequalities on complete manifolds”. *J. Math. Anal. Appl.* 409.1 (2014), pp. 84–90.

[126] Q. Yang, D. Su, and Y. Kong. “Hardy inequalities on Riemannian manifolds with negative curvature”. *Commun. Contemp. Math.* 16.2 (2014), p. 1350043.

[127] S.-T. Yin and Q. He. “The first eigenvalue of Finsler p-Laplacian”. *Differential Geom. Appl.* 35 (2014), pp. 30–49.

[128] L. Yuan and W. Zhao. “Cheeger’s constant and the first eigenvalue of a closed Finsler manifold”. *Preprint* (2013). arXiv: 1309.2115.

[129] L. Yuan, W. Zhao, and Y. Shen. “Improved Hardy and Rellich inequalities on nonreversible Finsler manifolds”. *J. Math. Anal. Appl.* 458.2 (2018), pp. 1512–1545.

[130] E. Zermelo. “Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung”. *Z. Angew. Math. Mech.* 11 (1931), pp. 114–124.

[131] E. Zermelo. “Über die Navigation in der Luft als Problem der Variationsrechnung”. *Jahresbericht der Deutschen Mathematiker-Vereinigung* 89 (1930), pp. 44–48.

[132] W. Zhao. “Hardy Inequalities with Best Constants on Finsler Metric Measure Manifolds”. *The Journal of Geometric Analysis* 31 (2021), pp. 1992–2032.