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Non-smooth elliptic problems on smooth manifolds

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Szeretnék köszönetet mondani szüleimnek amiért mindig is hittek bennem és lehetővé tették számomra, hogy idáig eljussak.

Végül, de nem utolsó sorban végtelenül hálás vagyok szerető feleségemnek, Juditnak, valamint gyermekeimnek, Enikőnek, Noéminek, Emmának és Ákosnak, amiért türelmükkel, kitartásukkal és szeretetükkel támogattak és ezzel lehetővé tették számomra, hogy a matematika egy részének rejtélyes világát felfedezhessem.

”Nature is written in mathematical language.”

Galileo Galilei

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Chapter 1

Introduction

The study of partial differential equations had already appeared within the analysis of physical models in the works of Euler, Lagrange and Laplace by the 18th century. Motivated by mathematical and physical problems, PDEs became an essential research area, both as standalone mathematical discipline as well as modeling various problems in physics, providing a bridge between pure mathematics and applications.

Arising in the context of several natural phenomena, PDEs have some well known, famous applications, like wave, Schrödinger, Maxwell, diffusion, Monge-Ampère and Navier-Stokes equations, respectively. The Laplace equation modeling the stationary state of the heat equation is the most simple variant of the elliptic class of PDEs besides the Poisson equation. The elliptic class of problems being generalization of the Laplace equation, is suitable to describe equilibrium states, or problems which are independent from the time. Mechanical or physical applications often induce not only continuous, but also discontinuous functions, where the idea is to ”fill the gaps” of the discontinuities with a set-valued generalized gradient of a locally Lipschitz function, e.g. von Kármán laminated plates problem, where the external force acts on adhesively connected laminated plates, analysed by Bocea, Panagiotopoulos and Rădulescu [9]. In this way, the appearance of non-smooth problems (thus, set-valued mappings) induces differential inclusions rather than differential equations.

Elliptic PDEs are usually studied on Sobolev spaces combined with powerful variational methods. Analyzing some fine properties of the energy functional associated to the studied problem, and exploiting variational methods like minimax or minimization principles, we may find critical points and prove in this way existence, uniqueness and multiplicity results. In case of discontinuous functions, non-smooth variational methods should be applied.

The primary objective of the thesis is to present recent research results in the study of elliptic differential inclusions. Applying recent geometrical researches, we show how to apply variational methods not only on Euclidean spaces but also on curved cases.

The thesis is based on the following papers:

(i) A. Kristály, I.I. Mezei and K. Szilák. *Differential inclusions involving oscillatory term.* Nonlinear Analysis, 197 (2020), 111834. [**D1** publication]

(ii) K. Szilák. *A non-smooth Neumann problem on compact Riemannian manifolds.* SACI 2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics.

(iii) A. Kristály, I.I. Mezei and K. Szilák. *Elliptic differential inclusions on non-compact Riemannian manifolds.* Nonlinear Analysis-Real World Applications, 69 (2023), 103740. [**D1** publication]

(iv) K. Szilák. *Schrödinger-Maxwell differential inclusion system.* SACI 2023 IEEE 17th International Symposium on Applied Computational Intelligence and Informatics.

(v) Á. Mester, K. Szilák, A Dirichlet inclusion problem on Finsler manifolds, CINTI 2023, IEEE 23rd International Symposium on Computational Intelligence and Informatics, November 20-22, 2023, Budapest, Hungary.

In the sequel a brief overview follows about the chapters. The thesis contains six chapters. Chapter 2 is devoted to present those results and notations which are indispensable in our investigations. In more details, this chapter gives a brief introduction into the calculus of locally Lipschitz functions, Riemannian manifolds, Sobolev spaces, functional inequalities and spectral estimates.

In Chapter 3, motivated by mechanical problems – where the external forces are non-smooth – we study an elliptic inclusion problem with a non-smooth oscillatory and a non-smooth, generic, p-order perturbation function in two settings. First, we consider the case when the oscillatory term oscillates near to the origin and the perturbation is of order $p > 0$ at origin. Applying various non-smooth variational methods, we provide a quite complete picture about the number of distinct, non-trivial weak solutions for the studied problem, depending on parameters p, λ and k, and we also prove a novel competition phenomena. As a counterpart, we also prove similar results whenever the nonlinear term oscillates at infinity and the perturbation is of order $p > 0$ at infinity. This chapter is based on the paper by Kristály, Mezei and Szilák [27].

In Chapter 4, considering a non-smooth elliptic problem on Riemannian manifolds, we discuss a differential inclusion, as a new application of a recent non-smooth Ricceri-type result. We prove that the studied inclusion problem has at least three distinct weak solutions whose norms are controlled whenever a suitable perturbation occurs. This chapter is based on Szilák [48].

Chapter 5 is devoted to focus onto a broad class of curved spaces. More precisely, we consider both Cartan-Hadamard manifolds and non-compact Riemannian manifolds with non-negative Ricci curvature. Within these geometric settings, we study an elliptic inclusion problem involving a singular term and a non-smooth nonlinearity, by proving various non-existence and existence results. In particular, four non-trivial G-invariant weak solutions are established in the above two settings (where G is a certain subgroup of isometries of the Riemannian manifold). In the first case, the nonlinear term is sub-quadratic, meanwhile in the second case it is super-quadratic at infinity. It turns out that the usual variational methods cannot be applied due to the lack of compactness, which will be recovered by isometric actions, combined with the principle of symmetric criticality. This chapter is based on the paper by Kristály, Mezei and Szilák [28].

In Chapter 6, motivated by physical problems, we consider a Schrödinger-Maxwell inclusion system involving a non-linear term, which is superlinear at the origin and sub-linear at infinity. Similarly to Chapter 5, we again focus on Cartan-Hadamard manifolds and non-compact Riemannian manifolds with non-negative Ricci curvature, respectively. Introducing a ”single variable” energy functional, we prove a non-existence result whenever the parameter λ is small enough, and by compensating the lack of compactness with isometric actions, we establish two non-trivial weak solutions for the inclusion system whenever the parameter λ is large enough. This chapter is based on Szilák [49].

Chapter 2

Preliminaries

2.1 Non-smooth analysis

When we are going to work with non-smooth functions, the classical analysis cannot be used. A wide class of such functions is provided by locally Lipschitz functions on Banach spaces, whose calculus has been developed mainly by Clarke [12]. In particular, the classical Gateaux derivative should be replaced by the generalized directional derivative in the sense of Clarke. This section is devoted to recall all these notions which will play fundamental role in our further investigations. Let X be a real Banach space with the norm $\|\cdot\|$.

Definition 2.1.1. *A function $f : X \to \mathbb{R}$ is locally Lipschitz if every point $u \in X$ possesses a neighborhood $U \subset X$ such that*

$$|f(u_1) - f(u_2)| \leq K\|u_1 - u_2\|, \quad \forall u_1, u_2 \in U, \tag{2.1}$$

for a constant $K > 0$ depending on U.

Definition 2.1.2. *The generalized directional derivative of the locally Lipschitz function $f : X \to \mathbb{R}$ at $u \in X$ in the direction $v \in X$ is given by*

$$f^0(u;v) := \limsup_{\substack{w \to u \\ t \searrow 0}} \frac{f(w+tv) - f(w)}{t}.$$

If $f : X \to \mathbb{R}$ is a function of class C^1 on X, then $f^0(u;v) = \langle f'(u), v\rangle$ for all $u, v \in X$. Hereafter, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_*$ stand for the duality mapping on (X^*, X) and the norm on X^*, respectively.

Definition 2.1.3. *The Clarke subdifferential $\partial f(u)$ of f at a point $u \in X$ is the subset of the dual space X^* given by*

$$\partial f(u) := \left\{ \xi \in X^* : \langle \xi, v \rangle \leq f^0(u; v),\ \forall v \in X \right\}.$$

Proposition 2.1.1. (Clarke [12]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. The following assertions hold*:

(i) *For every $u \in X$, $\partial f(u)$ is a nonempty, convex and weak*-compact subset of X^*. Moreover, $\|\xi\|_* \leq K$ for all $\xi \in \partial f(u)$, with $K > 0$ from* (2.1).

(ii) *For every $u \in X$, $f^0(u; \cdot)$ is the support function of $\partial f(u)$, i.e.,*

$$f^0(u; v) = \max \left\{ \langle \xi, v \rangle : \xi \in \partial f(u) \right\},\ \ \forall v \in X.$$

(iii) *The set-valued map $\partial f : X \rightsquigarrow X^*$ is weakly* closed. In particular, if X is finite dimensional, then ∂f is an upper semicontinuous set-valued map.*

(iv) *The function $(x, v) \mapsto f^\circ(x; v)$ is upper semicontinuous.*

(v) (Lebourg's mean value theorem) *Let U be an open subset of a Banach space X and u, v be two points of U such that the line segment $[u, v] = \{(1-t)u + tv : 0 \leq t \leq 1\} \subset U$. If $f : U \to \mathbb{R}$ is a Lipschitz function, then there exist $w \in (u, v)$ and $\xi \in \partial f(w)$ such that $f(v) - f(u) = \langle \xi, v - u \rangle$.*

(vi) *If $g : X \to \mathbb{R}$ is of class C^1 on X, then $\partial(g + f)(u) = g'(u) + \partial f(u)$ and $(g + f)^0(u; v) = \langle g'(u), v \rangle + f^0(u; v)$ for every $u, v \in X$.*

(vii) *$(-f)^0(u; v) = f^0(u; -v)$ for every $u, v \in X$.*

(viii) *$\partial(sf)(u) = s\partial f(u)$ for every $s \in \mathbb{R}$ and $u \in X$.*

(ix) *Chain rule: let us consider the composite function $f = g \circ h$ where $h : X \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given functions. Let denote h_i, $i \in \{1, ..., n\}$ be the component functions of h. We assume h_i is locally Lipschitz near x and g is too near $h(x)$. Then f is locally Lipschitz near x as well. Let us denote by α_i the elements of ∂g, and let $\alpha = (\alpha_1, ..., \alpha_n)$; then*

$$\partial f(x) \subset \overline{\text{co}}\{\sum \alpha_i \xi_i : \xi_i \in \partial h_i(x), \alpha \in \partial g(h(x))\},$$

where $\overline{\text{co}}$ denotes the weak-closed convex hull.

2.2 Riemannian geometry

Riemannian geometry studies smooth manifolds equipped with a family of inner products. Assigning an inner product to each tangent space on a smooth way, the objective of the Riemannian geometry is to understand deep relationships between distance, volume, curvature, geodesics, Jacobi fields, exponential maps, etc.; for comprehensive materials, see e.g. Jost [20] and Lee [34].

Definition 2.2.1. *Let M be a smooth manifold. T_xM denotes the tangent space at $x \in M$, and $TM = \bigcup_{x\in M} T_xM$ is the tangent bundle.*

Definition 2.2.2. *Let M be a smooth manifold. If M is endowed with a correspondence g which assigns an inner product (i.e. symmetric, bilinear, positive-definite form) $g_x : T_x(M) \times T_x(M) \to \mathbb{R}$ to each tangential space T_xM at $x \in M$ such that the mappings $g_{ij}(x) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \rangle$ is of class C^∞, then the metric g is called Riemannian metric. In this case, (M, g) is a Riemannian manifold.*

Definition 2.2.3. *Let (M, g) be an n-dimensional Riemannian manifold and $x \in M$. If $u, v \in T_xM$ are two linearly independent vectors of T_xM, then for the subspace S, spanned by u and v, the sectional curvature is defined by*

$$\mathbf{K}(S) = \frac{Rm_x(v, w, w, v)}{|v \wedge w|},$$

where Rm_x stands for the curvature tensor.

The sectional curvature **K** of (M, g) is bounded from below if for all $x \in M$ and $u, v \in T_xM$ there exists some $c \in \mathbb{R}$ such that $\mathbf{K}(u, v) \geq c$.

Definition 2.2.4. *Let (M, g) be an n-dimensional Riemannian manifold. Assuming that $e_1, \ldots, e_n$ is an orthonormal system of T_xM, the Ricci curvature in the direction $v = e_1 \in T_xM$ is defined by*

$$Ric_x(v) = \sum_{i=2}^{n} \mathbf{K}(e_i, v).$$

The Ricci curvature Ric of (M, g) is bounded from below if for all $x \in M$ and $v \in T_xM$ there exist some $c \in \mathbb{R}$, such that $Ric_x(v) \geq c$; in this case we denote $\mathsf{Ric}_{(M,g)} \geq c$.

Definition 2.2.5. *Let $d_g(x, y)$ be the Riemannian distance function associated to the Riemannian metric g. The open geodesic ball with center $x \in M$ and radius $r > 0$ is defined by $B_x(r) = \{y \in M : d_g(x, y) < r\}$.*

Let (M, g) be a Riemannian manifold, and k is an integer. The $k-th$ covariant derivatives of the function $u \in C^\infty(M)$ is denoted by $\nabla_g^k u$ (with $\nabla_g^0 u = u$).

The Laplace-Beltrami operator defined by $\triangle_g u = div(\nabla_g u)$ can be expressed in the local coordinates $(x^0, \ldots, x^n)$ as

$$\triangle_g u = g^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial u}{\partial x^k}.$$

The volume of the subset $D \subset M$ is given by $V_g(D) = \int_D 1 \mathrm{d}v_g$ in (M, g), where $\mathrm{d}v_g$ is the canonical volume element of (M, g).

Definition 2.2.6. *If* $\mathsf{Ric}_{(M,g)} \geq 0$*, the asymptotic volume ratio is given by*

$$\mathsf{AVR}_{(M,g)} = \lim_{r\to\infty}\frac{V_g(B_x(r))}{\omega_n r^n},$$

where $\omega_n = \pi^{n/2}/\Gamma(1 + n/2)$ *is the volume of the Euclidean unit ball in* $\mathbb{R}^n$*.*

Due to the Bishop-Gromov comparison principle, the asymptotic volume ratio is well-defined, and $\mathsf{AVR}_{(M,g)} \in [0, 1]$ provides deep geometric information about the manifold; for instance, $\mathsf{AVR}_{(M,g)} = 1$ if and only if (M, g) is isometric to the Euclidean space $\mathbb{R}^n$. Quantitatively speaking, closer value of $\mathsf{AVR}_{(M,g)}$ to 1 implies topologically closer manifold (M, g) to the Euclidean space $\mathbb{R}^n$, expressed in terms of the trivialization of higher homotopy groups of M, see Munn [40].

2.3 Sobolev spaces

Sobolev spaces play an important role in PDEs, allowing us to study weak solutions of differential equations, even when there is no solution in the classical sense.

Before defining the Sobolev spaces, we recall the notion of weak derivatives.

Definition 2.3.1. *Let assume that* $\Omega \subset \mathbb{R}^n$*.* $D^\alpha u \in L^1_{loc}(\Omega)$ *denotes the* α^{th} *weak derivative of the function* $u \in L^1_{loc}(\Omega)$*, if for all test functions* $v \in C_0^\infty(\Omega)$

$$\int_\Omega u(x) D^\alpha v(x) dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u(x) v(x) dx,$$

where $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}}$ *and* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *stands for multi-indexes with the notation* $|\alpha| := \sum_{i=1}^n \alpha_i$*.*

Definition 2.3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open domain, k be a non-negative integer, and $1 \leq p \leq \infty$, the Sobolov space is defined by the set*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq k\}.$$

The $W_0^{k,p}$ is the closure of C_0^∞ in the space $W^{k,p}$. The Sobolev space $W^{k,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, the norm is the usual sup-norm. When $p = 2$, the Sobolev space $W^{k,2}(\Omega)$ is a Hilbert space, denoted by $H^k(\Omega)$, equipped with the scalar product

$$\langle u, v \rangle_{H^k} = \sum_{k \leq |\alpha|} \int_\Omega D^\alpha u D^\alpha v dx.$$

In particular, the Sobolev space H^1 has the norm

$$\|u\|_{H^1} = \left(\int_\Omega u^2 + (\nabla u)^2 dx \right)^{\frac{1}{2}}$$

and scalar product

$$\langle u, v \rangle_{H^1} = \int_\Omega uv + \nabla u \nabla v dx.$$

Note that the Sobolev space is a Banach space, being reflexive whenever $1 < p < \infty$.

2.3.1 Sobolev spaces on Riemannian manifolds

Definition 2.3.3. *Let (M, g) be a Riemannian manifold. With a non-negative integer k and any $1 \leq p \leq \infty$, the Sobolev space $H^{k,p}(M)$ is the closure of the space*

$$C^{k,p} = \{u \in C^\infty(M) : \forall j = 0, \ldots, k, \int_M |\nabla_g^j u|^p \mathrm{d}v_g < \infty\}$$

with respect to the norm

$$\|u\|_{H^{k,p}} = \left(\sum_{j=0}^{k} \int_\Omega |\nabla_g^j u|^p \mathrm{d}v_g \right)^{\frac{1}{p}}.$$

When $p = 2$, the space $W^{k,2}(M)$ is a Hilbert space, denoted by $H^k(M)$, which is equipped with the scalar product in local coordinates

$$\langle u, v \rangle_{H^1} = \sum_{l=0}^{k} \int_M g^{i_1 j_1} \ldots g^{i_l j_l} (\nabla_g^l u)_{i_1 \ldots i_l} (\nabla_g^l v)_{j_1 \ldots j_l} \mathrm{d}v_g.$$

2.4 Variational principles

Variational calculus plays a central role in many physical phenomena. It is not only a powerful tool in the optimization of functionals, but also provides numerous principles and theorems to treat smooth and non-smooth PDEs. Since we are dealing with non-smooth problems, the variational calculus becomes a basic tool in our thesis.

Hereafter we collect those principles and theorems, which are required in our studies. For a comprehensive treatment, see Kristály, Rădulescu, Varga [31] and Chang [11].

In the sequel let X be a Banach space. We start with the definition of the critical point.

Definition 2.4.1. *Let $F : X \to \mathbb{R}$ be a locally Lipschitz functional. A point $x \in X$ is said to be a critical point of F, if $0 \in \partial F(x)$.*

In the sequel we recall crucial compactness conditions which are used to guarantee critical points in variational calculus. Before doing this, for the locally Lipschitz function F, we define $m(x) = \min\{\|\xi\|_* : \xi \in \partial F(x)\}$.

Definition 2.4.2. *The locally Lipshitz function $F : X \to \mathbb{R}$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (in short $(\mathrm{PS})_c$), if every sequence $\{x_n\} \subset X$ such that $F(x_n) \to c$ and $m(x_n) \to 0$, possesses a convergent subsequence.*

Definition 2.4.3. *The locally Lipschitz $F : X \to \mathbb{R}$ satisfies the Cerami condition at level c (in short $(\mathrm{C})_c$), if every sequence $\{x_n\} \subset X$ such that $F(x_n) \to c$ and $(\|x_n\| + 1)m(x_n) \to 0$, possesses a convergent subsequence.*

Remark 2.4.1. It is clear that $(\mathrm{PS})_c$ implies $(\mathrm{C})_c$.

Theorem 2.4.1. *Let X be reflexive Banach space. If the locally Lipschitz function $F : X \to \mathbb{R}$ satisfies the $(\mathrm{PS})_c$ condition with level c and is bounded from below, then $c = \inf_X F$ is a critical value of the function F, i.e., there exists $x \in X$ such that $0 \in \partial F(x)$ and $F(x) = c$.*

A non-smooth version of the famous Mountain Pass Theorem, initially established by Ambrosetti and Rabinowitz [2], can be stated as follows:

Theorem 2.4.2. (Mountain Pass Theorem) *Let X be a Banach space and $F : X \to \mathbb{R}$ be a locally Lipschitz function. We assume that there exist $x_1 \in X$, $\rho > 0$ and $\alpha > 0$ such* *that* $\|x_1\| > \rho$*,* $F(0) \leq 0$ *and*

$$\inf_{\|x\|=\rho} F(x) \geq \alpha > F(x_1).$$

If F *satisfies the* $(\mathrm{C})_c$ *condition at level*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} F(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1]; X) : \gamma(0) = 0,\ \gamma(1) = x_1\},$$

then c *is a critical value of* F*, and* $c \geq \alpha$*.*

Various forms of Theorem 2.4.2 is known in the literature, due to Kristály, Motreanu and Varga [29], Motreanu and Panagiotopoulos [39], etc.

Beside the above Mountain Pass Theorem, which guarantees a critical point, we shall use a Ricceri-type multiplicity theorem for locally Lipschitz functions, see Kristály, Marzantowicz and Varga [25]; its original form for C^1 functions can be found in Ricceri [44]:

Theorem 2.4.3. (Non-smooth Ricceri's multiplicity theorem) *Let* $(X, \|\cdot\|)$ *be a real Banach space,* X_1 *and* X_2 *be two Banach spaces such that embeddings* $X \hookrightarrow X_1$ *and* $X \hookrightarrow X_2$ *are compact. Let* Λ *be a real interval,* $h : [0,\infty) \to [0,\infty)$ *be a non-decreasing convex function and assume we have given two locally Lipschitz functions* $\phi_1 : X_1 \to \mathbb{R}$*,* $\phi_2 : X_2 \to \mathbb{R}$ *such that the locally Lipschitz function* $E_{\lambda,\mu} : X \to \mathbb{R}$*,* $E_{\lambda,\mu} = h(\|\cdot\|) + \lambda\phi_1 + \mu g \circ \phi_2$ *satisfies the* $(\mathrm{PS})_c$ *condition for every* $c \in \mathbb{R}$*,* $\lambda \in \Lambda$*,* $\mu \in [|\lambda| + 1]$ *and* $g \in \mathcal{G}_\tau$*,* $\tau \geq 0$*, where* $\mathcal{G}_\tau = \{f \in C^1(\mathbb{R},\mathbb{R}) | f \text{ is bounded, and } f(t) = t \text{ for any } t \in [-\tau,\tau]\}$*. Assume that* $h(\|\cdot\|) + \lambda\phi_1$ *is coercive on* X *for all* $\lambda \in \Lambda$ *and there exists* $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda\in\Lambda} \inf_{x\in X}[h(\|\cdot\|) + \lambda(\phi_1(x) + \rho)] < \inf_{x\in X} \sup_{\lambda\in\Lambda}[h(\|\cdot\|) + \lambda(\phi_1(x) + \rho)],$$

then there exists a non-empty open set $A \subset \Lambda$ *and* $r > 0$ *with the property that for every* $\lambda \in A$ *there exists* $\mu_0 \in [|\lambda| + 1]$ *such that, for each* $\mu \in [0, \mu_0]$ *the functional* $\mathcal{E}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\phi_1 + \mu\phi_2$ *has at least three critical points in* X *whose norms are less than* r*.*

2.4.1 Elliptic PDEs

One can prove that weak solutions of certain elliptic PDEs with boundary value constraints coincide with the critical points of the energy functional associated to the problem. We provide a simple (didactic) example to support this fact, which will be used as a guide in our further studies.

Example 2.4.1. Let $\Omega \subset \mathbb{R}^n$ be an open domain, and we consider the following Dirichlet problem

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{$\mathcal{P}_0$}$$

where Δ denotes the usual Laplace operator, and F is a locally Lipschitz function. The natural energy functional $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ associated to the problem $(\mathcal{P}_0)$ is defined as

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 - \int_\Omega F(u(x))dx.$$

It turns out that $\mathcal{E}$ is well-defined and locally Lipschitz. Additionally, if $u \in H_0^1$ is a critical point, then $0 \in \partial\mathcal{E}(u)$. In particular, for every $v \in H_0^1(\Omega)$ such that the mapping $x \mapsto \xi_x v \in L^1(\Omega)$, $\xi_x \in \partial F(u)$, we have

$$\int_\Omega \nabla u(x)\nabla v(x)dx - \int_\Omega \xi_x(x)v(x)dx = 0. \tag{2.2}$$

Applying the Green's theorem with the boundary condition it follows that

$$\int_\Omega \nabla u(x)\nabla v(x)dx = -\int_\Omega \Delta u(x)v(x)dx. \tag{2.3}$$

At this point we may integrate 2.3 into 2.2, and obtain that

$$-\int_\Omega \Delta u(x)v(x)dx = \int_\Omega \xi_x(x)v(x)dx \tag{2.4}$$

for all test function $v \in H_0^1(\Omega)$, which means indeed that u is a weak solution of the problem $(\mathcal{P}_0)$.

One can readily observe that, as we mentioned in the introduction, studying PDEs can be reduced to find critical points of the energy functional associated to the problem. In particular, analyzing coercivity, Palais-Smail condition, lower semicontinuity of the energy functional, existence and multiplicity results can be established by variational calculus (e.g. minimax principle or direct minimization arguments).

Sobolev compact embeddings can be exploited to prove the Palais-Smale or Cerami compactness condition. However, in the case when we are dealing with non-compact settings, this direct machinery is not working. In such cases, the lack of compactness has to be compensated with certain isometric actions together with the principle of symmetric criticality, see 2.4.3. In the sequel, we recall these notions/results.

2.4.2 Isometries

Let (M, g) be a Riemannian manifold and $\mathsf{Isom}_g(M)$ be the *group of isometries* of (M, g). Let us assume that G is a connected subgroup of $\mathsf{Isom}_g(M)$, and let

$$\mathrm{Fix}_M(G) = \{x \in M : \sigma(x) = x, \forall \sigma \in G\}$$

be the set of fixed points of the isometry group G in (M, g). The *G-orbit* of a point $x \in M$ is $\mathcal{O}_G^x = \{\sigma(x) : \sigma \in G\}$. The continuous action of the group G on M is *coercive* if for every $t > 0$ the set $\mathcal{O}_t := \{x \in M : \mathrm{diam}(\mathcal{O}_G^x) \leq t\}$ is bounded, see Skrzypczak and Tintarev [46, 47]; here $\mathrm{diam}(S)$ denotes the diameter of $S \subset M$. The action of G on $H^1(M)$ is defined by

$$(\sigma u)(x) = u(\sigma^{-1}(x)) \text{ for all } \sigma \in G, u \in H^1(M),\ x \in M,$$

where σ^{-1} is the inverse of the isometry σ. A function u is said to be radially symmetric with respect to the point $x_0 \in M$, if u depends on the Riemannian distance $d_g(x_0, \cdot)$.

It is standard to prove that G acts continuously and linearly on $H^1(M)$. For instance, if $\sigma_1, \sigma_2 \in G$, it turns out that for every $u \in H^1(M)$, $\sigma \in G$ and $x \in M$, we have

$$\begin{aligned}(\sigma_1 \circ \sigma_2)u(x) &= u((\sigma_1 \circ \sigma_2)^{-1}(x)) = u(\sigma_2^{-1}(\sigma_1^{-1}(x))) = (\sigma_2 u)(\sigma_1^{-1}(x)) \\ &= (\sigma_1(\sigma_2 u))(x).\end{aligned}$$

2.4.3 Principle of symmetric criticality.

Let G be a compact Lie group acting *linear isometrically* on the Banach space $(X, \|\cdot\|)$, i.e., the action $G \times X \to X$, $(\sigma, u) \mapsto \sigma u$ is continuous and for every $\sigma \in G$ the map $u \mapsto \sigma u$ is linear such that $\|\sigma u\| = \|u\|$ for every $u \in X$. Let

$$\mathsf{Fix}_X(G) = \{u \in X : \sigma u = u,\ \forall \sigma \in G\};$$

we notice that $\mathsf{Fix}_M(G)$ and $\mathsf{Fix}_X(G)$ should not be confused. A function $h : X \to \mathbb{R}$ is G-invariant, if $h(\sigma u) = h(u)$ for all $\sigma \in G$ and $x \in X$. According to Krawcewicz and Marzantowicz [22] (see also Costea, Kristály and Varga [13, Section 3.4]), the *principle of symmetric criticality* for locally Lipschitz functions can be stated as follows.

Proposition 2.4.1. (Krawcewicz and Marzantowicz [22]) *Let G be a compact Lie group acting linear isometrically on the real Banach space $(X, \|\cdot\|)$ and $h : X \to R$ be a G-invariant, locally Lipschitz functional. If $h|_G$ denotes the restriction of h to $\mathsf{Fix}_X(G)$ and $u \in \mathsf{Fix}_X(G)$ is a critical point of $h|_G$ then u is also a critical point of h.*

The smooth version of the principle of symmetric criticality has been provided by Palais [42] and later extended to various non-smooth settings.

2.5 Functional inequalities and spectral estimates

2.5.1 Cartan-Hadamard manifolds

Throughout this subsection, let (M, g) be an n-dimensional Cartan-Hadamard manifold (simply connected, complete Riemannian manifold with non-positive sectional curvature), $n \geq 3$.

Embeddings on Cartan-Hadamard manifold

We notice that in this geometric context, there exists $C_n > 0$ such that

$$\|u\|_{L^{2^*}} \leq C_n \left(\int_M |\nabla_g u|^2 \mathrm{d}v_g \right)^{1/2}, \ \ \forall u \in C_0^\infty(M),$$

see e.g. Hebey [19, Chapter 8], where $2^* = 2n/(n-2)$ is the critical Sobolev exponent. Moreover, the best Sobolev embedding constant C_n is precisely its Euclidean counterpart AT_n, provided by Aubin [3] and Talenti [50], whenever the Cartan-Hadamard conjecture holds on (M, g) (e.g. in dimensions 3 and 4). In high-dimensions, the sharp constant $C_n > 0$ is not known; however, a non-optimal form can be given by means of the Croke-constant as in Hebey [19, p. 239].

A density argument combined with a simple interpolation shows that the Sobolev space $H^1(M)$ is continuously embedded into $L^q(M)$ for every $q \in [2, 2^*]$; more precisely, there exists $K_q^- > 0$ such that

$$\|u\|_{L^q} \leq K_q^- \|u\|_{H^1}, \ \ \forall u \in H^1(M). \tag{2.5}$$

Hardy inequality

Let $x_0 \in M$ be fixed. The *Hardy inequality* holds on (M, g), which reads as

$$\frac{(n-2)^2}{4} \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g \leq \int_M |\nabla_g u|^2 \mathrm{d}v_g, \ \ \forall u \in H^1(M), \tag{2.6}$$

where $\frac{(n-2)^2}{4}$ is sharp and never achieved, see e.g. D'Ambrosio and Dipierro [15], and Kristály [24].

McKean's spectral gap

If the sectional curvature has the property $\mathbf{K} \leq -\kappa$ for some $\kappa > 0$, then *McKean's spectral gap theorem* asserts that

$$\gamma_{(M,g)} := \inf_{u \in H^1(M) \setminus \{0\}} \frac{\displaystyle\int_M |\nabla_g u|^2 \mathrm{d}v_g}{\displaystyle\int_M u^2 \mathrm{d}v_g} \geq \frac{(n-1)^2}{4} \kappa. \tag{2.7}$$

The inequality (2.7) is sharp, see e.g. on the n-dimensional hyperbolic space $\mathbb{H}_\kappa^n$ with constant sectional curvature $\mathbf{K} = -\kappa$; we also notice that the infimum in (2.7) is not achieved by any function $u \in H^1(M)$.

2.5.2 Riemannian manifolds with non-negative Ricci curvature

In this subsection we consider an n-dimensional $(n \geq 3)$ complete non-compact Riemannian manifold (M, g) with $\mathsf{Ric}_{(M,g)} \geq 0$.

Embeddings on Riemannian manifold with non-negative Ricci curvature

In the geometric context when (M, g) is a complete non-compact Riemannian manifold with $\mathsf{Ric}_{(M,g)} \geq 0$, a necessarily and sufficient condition to have the Sobolev embedding is the fact that $\mathsf{AVR}_{(M,g)} > 0$, see Coulhon and Saloff-Coste [14] and Hebey [19]. Moreover, a recent result of Balogh and Kristály [6] asserts that if $\mathsf{AVR}_{(M,g)} > 0$ then

$$\|u\|_{L^{2^*}} \leq \mathsf{AVR}_{(M,g)}^{-\frac{1}{n}} \mathsf{AT}_n \left(\int_M |\nabla_g u|^2 \mathrm{d}v_g \right)^{1/2}, \ \ \forall u \in H^1(M),$$

where the constant $\mathsf{AVR}_{(M,g)}^{-\frac{1}{n}}\mathsf{AT}_n$ is sharp; here AT_n stands for the best Sobolev embedding constant in the Euclidean Sobolev inequality on $\mathbb{R}^n$, see Aubin [3] and Talenti [50]. In particular, $H^1(M)$ is continuously embedded into $L^q(M)$ for every $q \in [2, 2^*]$; more precisely, there exists $K_q^+ > 0$ such that

$$\|u\|_{L^q} \leq K_q^+ \|u\|_{H^1}, \ \ \forall u \in H^1(M). \tag{2.8}$$

Hardy inequality

Given $x_0 \in M$ fixed, the *Hardy inequality* on (M, g) is verified as

$$\mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4} \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g \leq \int_M |\nabla_g u|^2 \mathrm{d}v_g, \ \ \forall u \in H^1(M), \tag{2.9}$$

see Kristály, Mester and Mezei [26]. The sharpness of the constant in (2.9) is not known unless we are in the classical Euclidean setting.

2.5.3 Embeddings on compact Euclidean spaces

Theorem 2.5.1. (Sobolev embedding theorem) *Let $\Omega \subset \mathbb{R}^n$ be an open set of class C^1. The embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous, whenever one of the following conditions hold for the real parameters p, q:*

(i) $1 \leq p < n$ *and* $p \leq q \leq p^*$;

(ii) $p = n$ *and* $p \leq q < \infty$;

(iii) $n < p \leq \infty$ *and* $q = \infty$,

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, the number p^* being the critical Sobolev exponent.

Theorem 2.5.2. (Rellich-Kondrachov theorem) *Let $\Omega \subset \mathbb{R}^n$ be of class C^1 and bounded. Then the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, whenever one of the following conditions holds for the real parameters p, q:*

(i) $1 \leq p < n$ *and* $1 \leq q \leq p^*$;

(ii) $p = n$ *and* $p \leq q < \infty$;

(iii) $n < p \leq \infty$ *and* $q = \infty$.

2.5.4 Embeddings on compact Riemannian manifolds

Theorem 2.5.3. *Let (M, g) be a compact Riemannian manifold of dimension n.*

(i) (Sobolev embedding theorem) *If $\frac{1}{q} \geq \frac{1}{p} - \frac{k}{n}$, then the embedding $H^{k,p} \hookrightarrow L^q$ is continuous.*

(ii) (Rellich-Kondrachov theorem) *If $\frac{1}{q} > \frac{1}{p} - \frac{k}{n}$, then the embedding $H^{k,p} \hookrightarrow L^q$ is compact.*

Theorem 2.5.4. *Let (M, g) be a compact n-dimensional Riemannian manifold.*

(i) *The embedding $H^{k,p}(M) \hookrightarrow L^q(M)$ is continuous if $p \leq q \leq \frac{np}{n-p}$ and compact whenever $p \leq q < \frac{np}{n-p}$.*

(ii) *If $\partial M \neq \emptyset$, the embedding $H^{k,p}(M) \hookrightarrow L^q(\partial M)$ is continuous if $p \leq q \leq \frac{p(n-1)}{n-p}$ and compact whenever $p \leq q < \frac{p(n-1)}{n-p}$.*

2.5.5 Embeddings on non-compact Riemannian manifolds

Let (M, g) be a complete, non-compact, n-dimensional Riemannian manifold and G be a compact connected subgroup of $\mathsf{Isom}_g(M)$. Recalling the main results of Farkas, Kristály and Mester [18], if one of the following curvature conditions hold

(i) (M, g) is a Cartan-Hadamard manifold and $\text{Fix}_M(G)$ is a singleton, or

(ii) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mathsf{AVR}_{(M,g)} > 0$ and G is coercive,

then the embedding $H^1_G(M) \hookrightarrow L^q(M)$ is compact for $q \in (2, 2^*)$.

We notice, that the embedding above is also compact for $q \in (2, 2^*)$ whenever the Ricci curvature is bounded from below and the injectivity radius is positive.

Part I

Differential inclusions - compact case

Chapter 3

Differential inclusions involving oscillatory terms

PDEs with perturbations that play central roles in physical and mechanical problems, have been subject of several investigations. Let consider the following elliptic PDE with perturbation

$$\begin{cases} -\Delta u(x) = f(u(x)) + \lambda g(u(x)), & x \in \Omega; \\ u \geq 0, & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where Δ is the usual Laplace operator, $\Omega \subset \mathbb{R}^n$ is a bounded open domain $(n \geq 2)$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying certain growth conditions at the origin and infinity, $g : \mathbb{R} \to \mathbb{R}$ is another continuous function which is going to compete with the original function f. When both functions f and g are of *polynomial type* of sub- and super-unit degree, the existence of at least one or two nontrivial solutions of (P_λ) is guaranteed, depending on the range of $\lambda > 0$, see e.g. Ambrosetti, Brezis and Cerami [1], Autuori and Pucci [4], de Figueiredo, Gossez and Ubilla [16]. In these papers variational arguments, sub- and super-solution methods as well as fixed point arguments are employed.

Another important class of problems of the type (P_λ) is studied whenever f has a certain *oscillation* (near the origin or at infinity) and g is a *perturbation*.

Although oscillatory functions seemingly call forth the existence of infinitely many solutions, it turns out that 'too classical' oscillatory functions do not have such a feature. Indeed, when $f(s) = c \sin s$ and $g = 0$, with $c > 0$ small enough, a simple use of the Poincaré inequality implies that problem (P_λ) has only the zero solution. However, when

f *strongly* oscillates, problem (P_λ) with 0 perturbation has indeed infinitely many different solutions; see e.g. Omari and Zanolin [41], Saint Raymond [45]. A novel competition phenomena for the case $g(s) = s^p$ $(s > 0)$ has been described for (P_λ) by Kristály and Moroşanu [30].

In mechanical applications, in turn, the perturbation may manifest in a *discontinuous* manner as a non-regular external force, see e.g. the gluing force in von Kármán laminated plates, cf. Bocea, Panagiotopoulos and Rădulescu [9], Motreanu and Panagiotopoulos [39] and Panagiotopoulos [43]. We consider the problem (P_λ) formulated into a more general form

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)), & x \in \Omega; \\ u \geq 0, & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{$\mathcal{D}_\lambda$}$$

where F and G are both non-smooth, locally Lipschitz functions having various growths, while ∂F and ∂G stand for the generalized gradients of F and G, respectively.

Extending the main results of Kristály and Moroşanu [30] we study the inclusion $(\mathcal{D}_\lambda)$ in two different settings, i.e., we analyze the number of distinct solutions of $(\mathcal{D}_\lambda)$ whenever ∂F oscillates near the origin/infinity and ∂G is of order $p > 0$ near the origin/infinity.

The organization of the present chapter is the following. In Section 3.1 we state our main assumptions and results, providing also some examples of functions fulfilling the assumptions. Section 3.2 contains a generic localization theorem for differential inclusions, while Sections 3.3 and 3.4 are devoted to the proof of our main results.

3.1 Main theorems

Let $F, G : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz functions and as usual, let us denote by ∂F and ∂G their generalized gradients in the sense of Clarke. Hereafter, $\mathbb{R}_+ = [0, \infty)$. Let $p > 0$, $\lambda \geq 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded open domain, and consider the elliptic differential inclusion problem

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)), & x \in \Omega; \\ u \geq 0, & x \in \Omega; \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{$\mathcal{D}_\lambda$}$$

The cases when ∂F oscillates near the *origin* or at *infinity* are studied in separated sections.

3.1.1 Oscillation near the origin

We assume that the beforementioned locally Lipschitz functions F and G satisfy the following conditions:

$(\mathbf{F}_0^0):\ F(0)=0;$

$(\mathbf{F}_1^0):\ -\infty < \liminf_{s\to 0^+}\frac{F(s)}{s^2};\ \limsup_{s\to 0^+}\frac{F(s)}{s^2}=+\infty;$

$(\mathbf{F}_2^0):\ l_0 := \liminf_{s\to 0^+}\frac{\max\{\xi:\xi\in\partial F(s)\}}{s}<0;$

$(\mathbf{G}_0^0):\ G(0)=0;$

$(\mathbf{G}_1^0):$ There exist $p>0$ and $\underline{c},\overline{c}\in\mathbb{R}$ such that

$$\underline{c}=\liminf_{s\to 0^+}\frac{\min\{\xi:\xi\in\partial G(s)\}}{s^p}\leq \limsup_{s\to 0^+}\frac{\max\{\xi:\xi\in\partial G(s)\}}{s^p}=\overline{c}.$$

Remark 3.1.1. Hypotheses $(\mathbf{F}_1^0)$ and $(\mathbf{F}_2^0)$ imply a strong oscillatory behavior of ∂F near the origin.

It is easy to prove that $0\in H_0^1(\Omega)$ is a solution of the differential inclusion $(\mathcal{D}_\lambda)$.

In the sequel we present a continuous oscillatory function and a locally Lipschitz function satisfying assumptions $(\mathbf{F}_0^0)-(\mathbf{F}_2^0)$ and $(\mathbf{G}_0^0)-(\mathbf{G}_1^0)$, respectively:

Example 3.1.1. Let us consider $F_0(s)=\int_0^s f_0(t)dt,\ s\geq 0$, where $f_0(t)=\sqrt{t}(\frac{1}{2}+\sin t^{-1}),\ t>0$ and $f_0(0)=0$, or some of its jumping variants. One can prove that $\partial F_0=f_0$ verifies the assumptions $(\mathbf{F}_0^0)-(\mathbf{F}_2^0)$. For a fixed $p>0$, let $G_0(s)=\ln(1+s^{p+2})\max\{0,\cos s^{-1}\},\ s>0$ and $G_0(0)=0$. It is clear that G_0 is not of class C^1 and verifies $(\mathbf{G}_1^0)$ with $\underline{c}=-1$ and $\overline{c}=1$, respectively; see Figure 3.1 representing both f_0 and G_0 (for $p=2$).

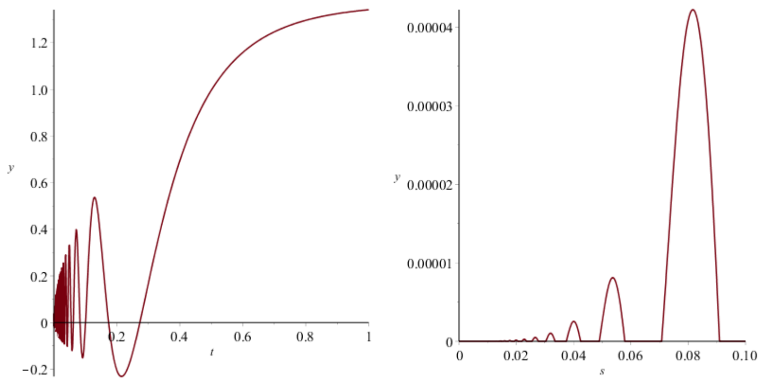

Figure 3.1: Graphs of f_0 and G_0 around the origin, respectively.

In what follows, we provide a quite complete picture about the competition concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we are going to show that when $p \geq 1$, then the 'leading' term is the oscillatory function ∂F; roughly speaking, one can say that the effect of $s \mapsto \partial G(s)$ is negligible in this competition. More precisely, we prove the following result.

Theorem 3.1.1. (Kristály, Mezei and Szilák [27]) (Case $p \geq 1$) *Assume that* $p \geq 1$ *and the locally Lipschitz functions* $F, G : \mathbb{R}_+ \to \mathbb{R}$ *satisfy* $(\mathbf{F}_0^0) - (\mathbf{F}_2^0)$ *and* $(\mathbf{G}_0^0) - (\mathbf{G}_1^0)$*. If* (i) *either* $p = 1$ *and* $\lambda \bar{c} < -l_0$ *(with* $\lambda \geq 0$*),* (ii) *or* $p > 1$ *and* $\lambda \geq 0$ *is arbitrary, then the differential inclusion problem* $(\mathcal{D}_\lambda)$ *admits a sequence* $\{u_i\}_i \subset H_0^1(\Omega)$ *of distinct weak solutions such that*

$$\lim_{i\to\infty} \|u_i\|_{H_0^1} = \lim_{i\to\infty} \|u_i\|_{L^\infty} = 0.$$

In the case when $p < 1$, the perturbation term ∂G may compete with the oscillatory function ∂F; we have the following theorem:

Theorem 3.1.2. (Kristály, Mezei and Szilák [27]) (Case $0 < p < 1$) *Assume* $0 < p < 1$ *and that the locally Lipschitz functions* $F, G : \mathbb{R}_+ \to \mathbb{R}$ *satisfy* $(\mathbf{F}_0^0) - (\mathbf{F}_2^0)$ *and* $(\mathbf{G}_0^0) - (\mathbf{G}_1^0)$*. Then, for every* $k \in \mathbb{N}$*, there exists* $\lambda_k > 0$ *such that the differential inclusion* $(\mathcal{D}_\lambda)$ *has at least* k *distinct weak solutions* $\{u_{1,\lambda}, ..., u_{k,\lambda}\} \subset H_0^1(\Omega)$ *whenever* $\lambda \in [0, \lambda_k]$*. Moreover,*

$$\|u_{i,\lambda}\|_{H_0^1} < i^{-1} \ \ and \ \ \|u_{i,\lambda}\|_{L^\infty} < i^{-1} \ \ for\ any\ i = \overline{1,k};\ \lambda \in [0, \lambda_k]. \tag{3.1}$$

3.1.2 Oscillation at infinity

We assume that the beforementioned locally Lipschitz functions F and G satisfy the following conditions:

$(\mathbf{F}_0^\infty)$: $F(0) = 0$;

$(\mathbf{F}_1^\infty)$: $-\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2}$; $\limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty$;

$(\mathbf{F}_2^\infty)$: $l_\infty := \liminf_{s\to\infty} \frac{\max\{\xi:\xi\in\partial F(s)\}}{s} < 0$;

$(\mathbf{G}_0^\infty)$: $G(0) = 0$;

$(\mathbf{G}_1^\infty)$: There exist $p > 0$ and $\underline{c}, \overline{c} \in \mathbb{R}$ such that

$$\underline{c} = \liminf_{s\to\infty} \frac{\min\{\xi : \xi \in \partial G(s)\}}{s^p} \leq \limsup_{s\to\infty} \frac{\max\{\xi : \xi \in \partial G(s)\}}{s^p} = \overline{c}.$$

Remark 3.1.2. Hypotheses $(\mathbf{F}_1^\infty)$ and $(\mathbf{F}_2^\infty)$ imply a strong oscillatory behavior of the set-valued map ∂F at infinity.

In the sequel we present a continuous oscillatory function and a locally Lipschitz function satisfying assumptions $(\mathbf{F}_0^\infty) - (\mathbf{F}_2^\infty)$ and $(\mathbf{G}_0^\infty) - (\mathbf{G}_1^\infty)$ respectively:

Example 3.1.2. We consider $F_\infty(s) = \int_0^s f_\infty(t)dt$, $s \geq 0$, where $f_\infty(t) = \sqrt{t}(\frac{1}{2} + \sin t)$, $t \geq 0$, or some of its jumping variants; one has that F_∞ verifies the assumptions $(\mathbf{F}_0^\infty) - (\mathbf{F}_2^\infty)$. For a fixed $p > 0$, let $G_\infty(s) = s^p \max\{0, \sin s\}$, $s \geq 0$; it is clear that G_∞ is a typically locally Lipschitz function on $[0, \infty)$ (not being of class C^1) and verifies $(\mathbf{G}_1^\infty)$ with $\underline{c} = -1$ and $\overline{c} = 1$; see Figure 3.2 representing both f_∞ and G_∞ (for $p = 2$), respectively.

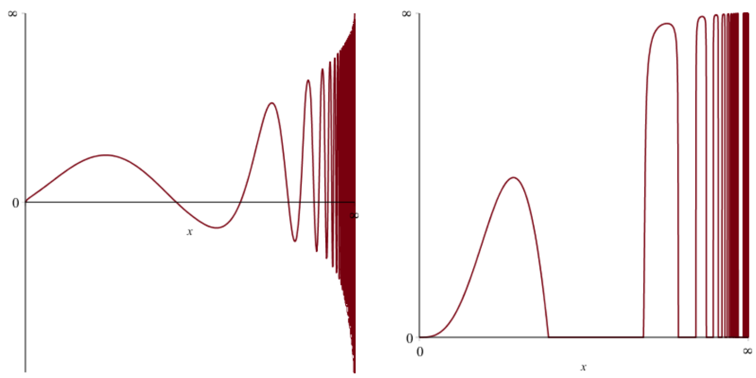

Figure 3.2: Graphs of f_∞ and G_∞ at infinity, respectively.

In the sequel, we investigate the competition at infinity concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we show that when $p \leq 1$ then the 'leading' term is the oscillatory function F, i.e., the effect of $s \mapsto \partial G(s)$ is negligible. More precisely, we prove the following result:

Theorem 3.1.3. (Kristály, Mezei and Szilák [27]) (Case $p \leq 1$) *Assume that $p \leq 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(\mathbf{F}_0^\infty) - (\mathbf{F}_2^\infty)$ and $(\mathbf{G}_0^\infty) - (\mathbf{G}_1^\infty)$. If*

(i) *either $p = 1$ and $\lambda \overline{c} \leq -l_0$ (with $\lambda \geq 0$),*

(ii) *or $p < 1$ and $\lambda \geq 0$ is arbitrary,*

then the differential inclusion $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i \to \infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{3.2}$$

Remark 3.1.3. Let us denote by 2^* the usual critical Sobolev exponent. In addition to (3.2), we also claim that $\lim_{i\to\infty} \|u_i^\infty\|_{H_0^1} = \infty$ whenever

$$\sup_{s \in [0,\infty)} \frac{\max\{|\xi| : \xi \in \partial F(s)\}}{1 + s^{2^*-1}} < \infty. \tag{3.3}$$

In the case when $p > 1$, it turns out that the perturbation term ∂G may compete with the oscillatory function ∂F; more precisely, we have the following theorem:

Theorem 3.1.4. (Kristály, Mezei and Szilák [27]) (Case $p > 1$) *Assume that $p > 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(\mathbf{F}_0^\infty) - (\mathbf{F}_2^\infty)$ and $(\mathbf{G}_0^\infty) - (\mathbf{G}_1^\infty)$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^\infty > 0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least k distinct weak solutions $\{u_{1,\lambda}, ..., u_{k,\lambda}\} \subset H_0^1(\Omega)$ whenever $\lambda \in [0, \lambda_k^\infty]$. Moreover,*

$$\|u_{i,\lambda}\|_{L^\infty} > i - 1 \ \ for\ any\ i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty]. \tag{3.4}$$

Remark 3.1.4. If the condition (3.3) holds and $p \leq 2^* - 1$ in Theorem 3.3, then we claim in addition that

$$\|u_{i,\lambda}^\infty\|_{H_0^1} > i - 1 \ \text{ for any } i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty].$$

3.2 Localization: a generic result

In this section we study the generalized form of the differential inclusion problem $(\mathcal{D}_\lambda)$, namely

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \ \ u(x) \geq 0, & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{$\mathcal{D}_A^k$}$$

where $k > 0$ and $A : [0, \infty) \to \mathbb{R}$ is a locally Lipschitz function with $A(0) = 0$ and

$(\mathbf{H}_A^1)$: there exists $M_A > 0$ such that

$$\max\{|\partial A(s)|\} := \max\{|\xi| : \xi \in \partial A(s)\} \leq M_A$$

for every $s \geq 0$;

$(\mathbf{H}_A^2)$: there are $0 < \delta < \eta$ such that $\max\{\xi : \xi \in \partial A(s)\} \leq 0$ for every $s \in [\delta, \eta]$.

For simplicity, let us extend the function A by $A(s) = 0$ for $s \leq 0$; the extended function is locally Lipschitz on the whole $\mathbb{R}$. The natural energy functional $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem $(\mathcal{D}_A^k)$ is defined by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega u^2 dx - \int_\Omega A(u(x))dx,$$

which is well-defined and locally Lipschitz on $H_0^1(\Omega)$.

Let us consider the number $\eta \in \mathbb{R}$ from $(\mathbf{H}_A^2)$ and the set

$$W^\eta = \{u \in H_0^1(\Omega) : \|u\|_{L^\infty} \leq \eta\}.$$

Our localization result reads as follows (see [30, Theorem 2.1] for its smooth form):

Theorem 3.2.1. *Let $k > 0$ and assume that hypotheses $(\mathbf{H}_A^1)$ and $(\mathbf{H}_A^2)$ hold. Then*

(i) *the energy functional $\mathcal{E}$ is bounded from below on W^η and its infimum is attained at some $\tilde{u} \in W^\eta$;*

(ii) *$\tilde{u}(x) \in [0, \delta]$ for a.e. $x \in \Omega$;*

(iii) *$\tilde{u}$ is a weak solution of the differential inclusion $(\mathcal{D}_A^k)$.*

Proof. (i) Using hypothesis $(\mathbf{H}_A^1)$ we have that the energy functional $\mathcal{E}$ is bounded from below on $H_0^1(\Omega)$. Moreover, due to the compactness of the Sobolev embedding, it turns out that $\mathcal{E}$ is sequentially weak lower semi-continuous on $H_0^1(\Omega)$. In addition, it is clear, that the set W^η is weakly closed, being convex and closed in $H_0^1(\Omega)$. Thus, there is $\tilde{u} \in W^\eta$ which is a minimum point of $\mathcal{E}$ on the set W^η, see Zeidler [51].

(ii) We introduce the set $L = \{x \in \Omega : \tilde{u}(x) \notin [0, \delta]\}$ and suppose indirectly that the measure $m(L) > 0$. We define the functions $\gamma(s) := \min(\max(s, 0), \delta)$ and $w := \gamma \circ \tilde{u}$, and claim that $w \in H_0^1(\Omega)$. Indeed, since $\gamma(0) = 0$ and γ is a Lipschitz function, the superposition theorem of Marcus and Mizel [37] implies that $w \in H_0^1(\Omega)$. By the definiton of γ, we have $0 \leq w(x) \leq \delta$ for a.e. Ω and combining $w \in H_0^1(\Omega)$ with assumption $(\mathbf{H}_A^2)$, $w \in W^\eta$ concludes.

Let us decompose the set L into L_1 and L_2,

$$L_1 = \{x \in L : \tilde{u}(x) < 0\} \text{ and } L_2 = \{x \in L : \tilde{u}(x) > \delta\}.$$

In particular, $L = L_1 \cup L_2$, and by definition, it follows that

$$w(x) = \begin{cases} \tilde{u}(x), & \text{for all } x \in \Omega \setminus L, \\ 0, & \text{for all } x \in L_1, \\ \delta, & \text{for all } x \in L_2. \end{cases}$$

Let us consider the expression

$$\begin{aligned} \mathcal{E}(w) - \mathcal{E}(\tilde{u}) &= \frac{1}{2}\left[\|w\|_{H_0^1}^2 - \|\tilde{u}\|_{H_0^1}^2\right] + \frac{k}{2}\int_\Omega \left[w^2 - \tilde{u}^2\right] dx - \int_\Omega [A(w(x)) - A(\tilde{u}(x))]dx \\ &= -\frac{1}{2}\int_L |\nabla \tilde{u}|^2 dx + \frac{k}{2}\int_L [w^2 - \tilde{u}^2]dx - \int_L [A(w(x)) - A(\tilde{u}(x))]dx. \end{aligned}$$

On account of $k > 0$, we have

$$k\int_L [w^2 - \tilde{u}^2]dx = -k\int_{L_1} \tilde{u}^2 dx + k\int_{L_2} [\delta^2 - \tilde{u}^2]dx \leq 0.$$

Taking into consideration that $A(s) = 0$ for all $s \leq 0$, we conclude

$$\int_{L_1} [A(w(x)) - A(\tilde{u}(x))]dx = 0.$$

By means of the Lebourg's mean value theorem, for a.e. $x \in L_2$, there exists $\theta(x) \in [\delta, \tilde{u}(x)] \subseteq [\delta, \eta]$ such that

$$A(w(x)) - A(\tilde{u}(x)) = A(\delta) - A(\tilde{u}(x)) = a(\theta(x))(\delta - \tilde{u}(x)),$$

where $a(\theta(x)) \in \partial A(\theta(x))$. Due to assumption $(\mathbf{H}_A^2)$, it turns out that

$$\int_{L_2} [A(w(x)) - A(\tilde{u}(x))]dx \geq 0.$$

Combining the above estimates, we obtain that $\mathcal{E}(w) - \mathcal{E}(\tilde{u}) \leq 0$. On the other side, since $w \in W^\eta$, claim (i) imply that $\mathcal{E}(w) \geq \mathcal{E}(\tilde{u}) = \inf_{W^\eta} \mathcal{E}$, thus every term in the difference $\mathcal{E}(w) - \mathcal{E}(\tilde{u})$ should be zero; in particular,

$$\int_{L_1} \tilde{u}^2 dx = \int_{L_2} [\tilde{u}^2 - \delta^2]dx = 0.$$

The latter relation implies in particular that $m(L) = 0$, which is a contradiction, completing the proof of (ii).

(iii) Since $\tilde{u}(x) \in [0, \delta]$ for a.e. $x \in \Omega$, an arbitrarily small perturbation $\tilde{u} + \epsilon v$ of $\tilde{u}$ with $0 < \epsilon \ll 1$ and $v \in C_0^\infty(\Omega)$ still implies that $\mathcal{E}(\tilde{u} + \varepsilon v) \geq \mathcal{E}(\tilde{u})$; accordingly, $\tilde{u}$ is a minimum point for $\mathcal{E}$ in the strong topology of $H_0^1(\Omega)$, thus $0 \in \partial\mathcal{E}(\tilde{u})$. Consequently, it follows that $\tilde{u}$ is a weak solution of the differential inclusion $(\mathcal{D}_A^k)$. □

Remark 3.2.1. In the sequel we need a truncation function of $H_0^1(\Omega)$, see also [30]. To construct this function, let $B(x_0, r) \subset \Omega$ be the n-dimensional ball with radius $r > 0$ and center $x_0 \in \Omega$. For $s > 0$, we introduce the function

$$w_s(x) = \begin{cases} 0, & \text{if } \quad x \in \Omega \setminus B(x_0, r); \\ s, & \text{if } \quad x \in B(x_0, r/2); \\ \frac{2s}{r}(r - |x - x_0|), & \text{if } \quad x \in B(x_0, r) \setminus B(x_0, r/2). \end{cases} \tag{3.5}$$

We observe that $w_s \in H_0^1(\Omega)$, $\|w_s\|_{L^\infty} = s$ and

$$\|w_s\|_{H_0^1}^2 = \int_\Omega |\nabla w_s|^2 dx = 4r^{n-2}(1 - 2^{-n})\omega_n s^2 \equiv C(r, n)s^2 > 0; \tag{3.6}$$

hereafter ω_n stands for the volume of $B(0, 1) \subset \mathbb{R}^n$.

3.3 Proof of Theorems 3.1.1 and 3.1.2

Before giving the proof of Theorems 3.1.1 and 3.1.2, in the first part of this section we study the following differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \;\; u(x) \geq 0, & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{$\mathcal{D}_A^k$}$$

where $k > 0$ and the locally Lipschitz function $A : \mathbb{R}_+ \to \mathbb{R}$ verifies

$(\mathbf{H}_0^0)$: $A(0) = 0$;

$(\mathbf{H}_1^0)$: $-\infty < \liminf_{s \to 0^+} \frac{A(s)}{s^2}$ and $\limsup_{s \to 0^+} \frac{A(s)}{s^2} = +\infty$;

$(\mathbf{H}_2^0)$: there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \eta_{i+1} < \delta_i < \eta_i$, $\lim_{i \to \infty} \eta_i = 0$, and

$$\max\{\partial A(s)\} := \max\{\xi : \xi \in \partial A(s)\} \leq 0$$

for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

Theorem 3.3.1. *Let $k > 0$ and assume hypotheses $(\mathbf{H}_0^0)$, $(\mathbf{H}_1^0)$ and $(\mathbf{H}_2^0)$ hold. Then there exists a sequence $\{u_i^0\}_i \subset H_0^1(\Omega)$ of distinct weak solutions of the differential inclusion problem $(\mathcal{D}_A^k)$ such that*

$$\lim_{i\to\infty} \|u_i^0\|_{H_0^1} = \lim_{i\to\infty} \|u_i^0\|_{L^\infty} = 0.$$

Proof. We may assume that $\{\delta_i\}_i$, $\{\eta_i\}_i \subset (0,1)$. For any fixed number $i \in \mathbb{N}$ we define the truncated locally Lipschitz function $A_i : \mathbb{R} \to \mathbb{R}$ by

$$A_i(s) = A(\tau_{\eta_i}(s)), \tag{3.7}$$

where $A(s) = 0$ for $s \le 0$ and $\tau_{\eta_i}(s) = \min(\eta_i, s)$.

Applying the truncated locally Lipschitz function $A_i(s)$, $i \in \mathbb{N}$ instead of $A(s)$ in $(\mathcal{D}_A^k)$, we introduce the problem $(\mathcal{D}_{A_i}^k)$, and for later usage one can associate the energy functional with that, namely $\mathcal{E}_i : H_0^1(\Omega) \to \mathbb{R}$, $i \in \mathbb{N}$.

We notice that for $s \ge 0$, recalling the chain rule, we have

$$\partial A_i(s) = \begin{cases} \partial A(s) & \text{if } \ s < \eta_i, \\ \overline{\text{co}}\{0, \partial A(\eta_i)\} & \text{if } \ s = \eta_i, \\ \{0\} & \text{if } \ s > \eta_i. \end{cases}$$

The fact that on the compact set $[0, \eta_i]$, the upper semicontinuous set-valued map $s \mapsto \partial A_i(s)$ attains its supremum imply that there exists $M_{A_i} > 0$ such that

$$\max |\partial A_i(s)| := \max\{|\xi| : \xi \in \partial A_i(s)\} \le M_{A_i}$$

for every $s \ge 0$, i.e., the assumption $(\mathbf{H}_{2,A_i}^0)$ holds. Applying $(\mathbf{H}_1^0)$ on $[\delta_i, \eta_i]$, $i \in \mathbb{N}$ implies that $(\mathbf{H}_{1,A_i}^0)$ is verified as well.

Accordingly, the assumptions of Theorem 3.2.1 are clearly verified for every $i \in \mathbb{N}$ with $[\delta_i, \eta_i]$, thus there exists $u_i^0 \in W^{\eta_i}$ such that

$$u_i^0 \text{ is the minimum point of the functional } \mathcal{E}_i \text{ on } W^{\eta_i}, \tag{3.8}$$

$$u_i^0(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega, \tag{3.9}$$

$$u_i^0 \text{ is a solution of } (\mathcal{D}_{A_i}^k).$$

Taking into account relations above, u_i^0 is a weak solution also for the differential inclusion problem $(\mathcal{D}_A^k)$.

What is still remaining is that there are infinitely many distinct elements in the sequence $\{u_i^0\}_i$. To conclude it, we first prove the following two lemmas:

Lemma 3.3.1. *If the assumptions of Theorem 3.3.1 hold, we have*

$$\mathcal{E}_i(u_i^0) < 0 \;\; for\ all \;\; i \in \mathbb{N}. \tag{3.10}$$

Proof. The left part of $(\mathbf{H}_1^0)$ implies the existence of some $l_0 > 0$ and $\zeta \in (0, \eta_1)$ such that

$$A(s) \geq -l_0 s^2 \text{ for all } s \in (0, \zeta). \tag{3.11}$$

We may have $L_0 > 0$ such that

$$\frac{1}{2}C(r,n) + \left(\frac{k}{2} + l_0\right) m(\Omega) < L_0 (r/2)^n \omega_n, \tag{3.12}$$

where $r > 0$ and $C(r,n) > 0$ come from (3.6). Based on the right part of $(\mathbf{H}_1^0)$, we can find a sequence $\{\tilde{s}_i\}_i \subset (0, \zeta)$ such that $\tilde{s}_i \leq \delta_i$ and

$$A(\tilde{s}_i) > L_0 \tilde{s}_i^2 \;\text{ for all } i \in \mathbb{N}. \tag{3.13}$$

Let $i \in \mathbb{N}$ be a fixed number and let $w_{\tilde{s}_i} \in H_0^1(\Omega)$ be the function from (3.5) corresponding to the value $\tilde{s}_i > 0$. Combining the fact that $w_{\tilde{s}_i} \in W^{\eta_i}$, with expressions (3.11), (3.13) and (3.6) we have

$$\begin{aligned}
\mathcal{E}_i(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega w_{\tilde{s}_i}^2 dx - \int_\Omega A_i(w_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,n)\tilde{s}_i^2 + \frac{k}{2}\int_\Omega w_{\tilde{s}_i}^2 dx - \int_{B(x_0,r/2)} A(\tilde{s}_i)dx \\
&- \int_{B(x_0,r)\setminus B(x_0,r/2)} A(w_{\tilde{s}_i}(x))dx \\
&\leq \left[\frac{1}{2}C(r,n) + \frac{k}{2}m(\Omega) - L_0(r/2)^n\omega_n + l_0 m(\Omega)\right]\tilde{s}_i^2.
\end{aligned}$$

Accordingly, with (3.8) and (3.12), we deduce that

$$\mathcal{E}_i(u_i^0) = \min_{W^{\eta_i}} \mathcal{E}_i \leq \mathcal{E}_i(w_{\tilde{s}_i}) < 0, \tag{3.14}$$

which proves the claim. $\square$

Lemma 3.3.2. *Under the assumptions of Theorem 3.3.1, we have*

$$\lim_{i\to\infty} \mathcal{E}_i(u_i^0) = 0 \;\; for\ all \;\; i \in \mathbb{N}. \tag{3.15}$$

Proof. For every $i \in \mathbb{N}$, combining the Lebourg's mean value theorem with relations (3.7), (3.9) and assumption $(\mathbf{H}_0^0)$, we can conclude that

$$\mathcal{E}_i(u_i^0) \geq -\int_\Omega A_i(u_i^0(x))dx = -\int_\Omega A_1(u_i^0(x))dx \geq -M_{A_1} m(\Omega)\delta_i.$$

Since $\lim_{i\to\infty}\delta_i = 0$, the latter estimate and (3.14) prove the claim (3.15). $\square$

Based on (3.7) and (3.9), we have that $\mathcal{E}_i(u_i^0) = \mathcal{E}_1(u_i^0)$ for all $i \in \mathbb{N}$. This relation with (3.10) and (3.15) implies that the sequence $\{u_i^0\}_i$ contains infinitely many distinct elements.

We now prove the last statement of the theorem. On one hand, combining the fact that (3.9) implies $\|u_i^0\|_{L^\infty} \le \delta_i$ for all $i \in \mathbb{N}$ with $\lim_{i\to\infty}\delta_i = 0$, $\lim_{i\to\infty}\|u_i^0\|_{L^\infty} = 0$ clearly follows. On the other hand, by using $k > 0$, (3.7), (3.9) and (3.14) we can conclude that

$$\begin{aligned}\frac{1}{2}\|u_i^0\|_{H_0^1}^2 &\le \frac{1}{2}\|u_i^0\|_{H_0^1}^2 + \frac{k}{2}\int_\Omega (u_i^0)^2dx < \int_\Omega A_i(u_i^0(x))dx = \int_\Omega A_1(u_i^0(x))dx \\ &\le M_{A_1}m(\Omega)\delta_i, \quad \text{for all } i \in \mathbb{N}.\end{aligned}$$

The latter expression with $\lim_{i\to\infty}\delta_i = 0$ imply that $\lim_{i\to\infty}\|u_i^0\|_{H_0^1} = 0$, which completes the proof of Theorem 3.3.1. $\square$

Proof of Theorem 3.1.1. We split the proof into two parts.

(i) *Case* $p = 1$. We assume that $\lambda \ge 0$ with $\lambda\bar{c} < -l_0$ and let us fix $\tilde{\lambda}_0 \in \mathbb{R}$ such that $\lambda\bar{c} < \tilde{\lambda}_0 < -l_0$. With these choices we define

$$k := \tilde{\lambda}_0 - \lambda\bar{c} > 0 \text{ and } A(s) := F(s) + \frac{\tilde{\lambda}_0}{2}s^2 + \lambda\left(G(s) - \frac{\bar{c}}{2}s^2\right) \text{ for every } s \in [0,\infty). \tag{3.16}$$

The fact that $A(0) = 0$ implies that the assumption $(\mathbf{H}_0^0)$ holds. Since $p = 1$, by assumption $(\mathbf{G}_1^0)$ we have

$$\underline{c} = \liminf_{s\to 0^+}\frac{\min\{\partial G(s)\}}{s} \le \limsup_{s\to 0^+}\frac{\max\{\partial G(s)\}}{s} = \bar{c}.$$

In particular, for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\max\{\partial G(s)\} - \bar{c}s < \epsilon s, \ \forall s \in [0,\gamma],$$

and

$$\min\{\partial G(s)\} - \underline{c}s > -\epsilon s, \ \forall s \in [0,\gamma].$$

For $s \in [0,\gamma]$, Lebourg's mean value theorem and $G(0) = 0$ imply that there exists $\xi_s \in \partial G(\theta_s s)$ for some $\theta_s \in [0,1]$ such that $G(s) - G(0) = \xi_s s$. Accordingly, for every $s \in [0,\gamma]$ we have that

$$(\underline{c} - \epsilon)s^2 \le G(s) \le (\bar{c} + \epsilon)s^2. \tag{3.17}$$

Combining (3.17) with the assumption (F_1^0) give us

$$\begin{aligned}\liminf_{s\to 0^+}\frac{A(s)}{s^2} &\geq \liminf_{s\to 0^+}\frac{F(s)}{s^2}+\frac{\tilde{\lambda}_0-\lambda\overline{c}}{2}+\lambda\liminf_{s\to 0^+}\frac{G(s)}{s^2}\\ &\geq \liminf_{s\to 0^+}\frac{F(s)}{s^2}+\frac{\tilde{\lambda}_0-\lambda\overline{c}}{2}+\lambda\underline{c}\\ &> -\infty\end{aligned}$$

and

$$\limsup_{s\to 0^+}\frac{A(s)}{s^2}\geq \limsup_{s\to 0^+}\frac{F(s)}{s^2}+\frac{\tilde{\lambda}_0-\lambda\overline{c}}{2}+\lambda\liminf_{s\to 0^+}\frac{G(s)}{s^2}=+\infty,$$

which imply that the assumption $(\mathbf{H}_1^0)$ follows.

The generalized gradient of the locally Lipschitz function A

$$\partial A(s)\subseteq \partial F(s)+\tilde{\lambda}_0 s+\lambda(\partial G(s)-\overline{c}s), \tag{3.18}$$

with $\lambda\geq 0$ give us

$$\max\{\partial A(s)\}\leq \max\{\partial F(s)+\tilde{\lambda}_0 s\}+\lambda\max\{\partial G(s)-\overline{c}s\}. \tag{3.19}$$

Since $p=1$, combining expression 3.19 with assumptions $(\mathbf{F}_2^0)$ and $(\mathbf{G}_1^0)$, the following estimation follows

$$\begin{aligned}\liminf_{s\to 0^+}\frac{\max\{\partial A(s)\}}{s} &\leq \liminf_{s\to 0^+}\frac{\max\{\partial F(s)\}}{s}+\tilde{\lambda}_0-\lambda\overline{c}+\lambda\limsup_{s\to 0^+}\frac{\max\{\partial G(s)\}}{s}\\ &\leq l_0+\tilde{\lambda}_0\\ &< 0.\end{aligned}$$

Therefore, we may have a sequence $\{s_i\}_i\subset(0,1)$ converging to 0 such that

$$\frac{\max\{\partial A(s_i)\}}{s_i}<0,$$

i.e., $\max\{\partial A(s_i)\}<0$ for all $i\in\mathbb{N}$. By using the upper semicontinuity of $s\mapsto\partial A(s)$, we may choose two numbers $\delta_i,\eta_i\in(0,1)$ with $\delta_i<s_i<\eta_i$ such that $\partial A(s)\subset\partial A(s_i)+[-\epsilon_i,\epsilon_i]$ for every $s\in[\delta_i,\eta_i]$, where $\epsilon_i:=-\max\{\partial A(s_i)\}/2>0$. In particular, we have $\max\{\partial A(s)\}\leq 0$ for all $s\in[\delta_i,\eta_i]$. Thus, we may fix two sequences $\{\delta_i\}_i,\{\eta_i\}_i\subset(0,1)$ such that $0<\eta_{i+1}<\delta_i<s_i<\eta_i$, $\lim_{i\to\infty}\eta_i=0$, and $\max\{\partial A(s)\}\leq 0$ for all $s\in[\delta_i,\eta_i]$ and $i\in\mathbb{N}$. Accordingly, the assumption $(\mathbf{H}_2^0)$ is verified as well, thus we are in the position now that we can apply Theorem 3.3.1 with the inclusion (3.18) and choices (3.16), i.e., there exists a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x) + (\tilde{\lambda}_0 - \lambda\overline{c})u_i(x) \in \partial F(u_i(x)) + \tilde{\lambda}_0 u_i(x) + \lambda(\partial G(u_i(x)) - \overline{c}u_i(x)), \\ x \in \Omega; \\ u_i(x) \geq 0, \quad x \in \Omega; \\ u_i(x) = 0, \quad x \in \partial\Omega. \end{cases}$$

In particular, u_i solves problem $(\mathcal{D}_\lambda)$, $i \in \mathbb{N}$, which completes the proof of (i).

(ii) *Case* $p > 1$. Let $\lambda \geq 0$ be arbitrary fixed and choose a number $\lambda_0 \in (0, -l_0)$. Let

$$k := \lambda_0 > 0 \text{ and } A(s) := F(s) + \lambda G(s) + \lambda_0 \frac{s^2}{2} \text{ for every } s \in [0, \infty). \tag{3.20}$$

We can observe that $F(0) = G(0) = 0$, thus the hypothesis $(\mathbf{H}_0^0)$ holds.

Since $p > 1$, assumption $(\mathbf{G}_1^0)$ gives us that,

$$\lim_{s\to 0^+} \frac{\min\{\partial G(s)\}}{s} = \lim_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} = 0. \tag{3.21}$$

In particular, for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that

$$\max\{\partial G(s)\} - \overline{c}s^p < \epsilon s^p, \ \forall s \in [0, \gamma]$$

and

$$\min\{\partial G(s)\} - \underline{c}s^p > -\epsilon s^p, \ \forall s \in [0, \gamma].$$

For a fixed $s \in [0, \gamma]$, by Lebourg's mean value theorem and $G(0) = 0$ we conclude again that $G(s) - G(0) = \xi_s s$. Accordingly, for sufficiently small $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that $(\underline{c} - \epsilon)s^{p+1} \leq G(s) \leq (\overline{c} + \epsilon)s^{p+1}$ for every $s \in [0, \gamma]$. Thus, since $p > 1$,

$$\lim_{s\to 0^+} \frac{G(s)}{s^2} = \lim_{s\to 0^+} \frac{G(s)}{s^{p+1}} s^{p-1} = 0.$$

Therefore, by using (3.20) and assumption ($\mathbf{F}_1^0$), we conclude that

$$\liminf_{s\to 0^+} \frac{A(s)}{s^2} = \liminf_{s\to 0^+} \frac{F(s)}{s^2} + \lambda \lim_{s\to 0^+} \frac{G(s)}{s^2} + \frac{\lambda_0}{2} > -\infty,$$

and

$$\limsup_{s\to 0^+} \frac{A(s)}{s^2} = \infty,$$

which means that $(\mathbf{H}_1^0)$ follows. Combining the generalized gradient of the locally Lipschitz function A,

$$\partial A(s) \subseteq \partial F(s) + \lambda \partial G(s) + \lambda_0 s, \tag{3.22}$$

and $\lambda \geq 0$, we have that

$$\max\{\partial A(s)\} \leq \max\{\partial F(s)\} + \max\{\lambda \partial G(s) + \lambda_0 s\}.$$

Combining expression 3.21 with assumptions ($\mathbf{F}_2^0$) and ($\mathbf{G}_1^0$), the following estimate follows

$$\begin{aligned}\liminf_{s\to 0^+} \frac{\max\{\partial A(s)\}}{s} &= \liminf_{s\to 0^+} \frac{\max\{\partial F(s)\}}{s} + \lambda \lim_{s\to 0^+} \frac{\max\{\partial G(s)\}}{s} + \lambda_0 \\ &= l_0 + \lambda_0 \\ &< 0,\end{aligned}$$

and the upper semicontinuity of ∂A implies the existence of two sequences $\{\delta_i\}_i$ and $\{\eta_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and $\max\{\partial A(s)\} \leq 0$ for all $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$; therefore, hypothesis ($\mathbf{H}_2^0$) follows. Now we are in the position that we can apply again Theorem 3.3.1 with the inclusion 3.22 and choises 3.20, i.e., there is a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x) + \lambda_0 u_i(x) \in \partial A(u_i(x)) \subseteq \partial F(u_i(x)) + \lambda \partial G(u_i(x)) + \lambda_0 u_i(x), \\ x \in \Omega; \\ u_i(x) \geq 0, \;\; x \in \Omega; \\ u_i(x) = 0, \;\; x \in \partial\Omega, \end{cases}$$

which means that u_i solves problem ($\mathcal{D}_\lambda$), $i \in \mathbb{N}$. This completes the proof of Theorem 3.1.1. □

Proof of Theorem 3.1.2. The proof is done in two steps:

(i) We use the same assumptions and definitions as in the proof of Theorem 3.1.1 (ii), i.e. we assume $\lambda_0 \in (0, -l_0)$, $\lambda \geq 0$ and similarly to the proof of Theorem 3.1.1, let us define

$$k := \lambda_0 > 0 \;\text{ and }\; A^\lambda(s) := F(s) + \lambda G(s) + \lambda_0 \frac{s^2}{2} \text{ for every } s \in [0,\infty). \tag{3.23}$$

The generalized gradient of the locally Lipschitz function A for every $s \geq 0$ is given by

$$\partial A^\lambda(s) \subseteq \partial F(s) + \lambda_0 s + \lambda \partial G(s).$$

On account of the assumption ($\mathbf{F}_2^0$), there is a sequence $\{s_i\}_i \subset (0,1)$ converging to 0 such that

$$\max\{\partial A^{\lambda=0}(s_i)\} \leq \max\{\partial F(s_i)\} + \lambda_0 s_i < 0.$$

Thus, due to the upper semicontinuity of $(s,\lambda) \mapsto \partial A^\lambda(s)$, we can choose three sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0,1)$ such that $0 < \eta_{i+1} < \delta_i < s_i < \eta_i$, $\lim_{i\to\infty} \eta_i = 0$, and

$$\max\{\partial A^\lambda(s)\} \leq 0 \text{ for all } \lambda \in [0,\lambda_i], s \in [\delta_i, \eta_i],\ i \in \mathbb{N}.$$

Without any loss of generality, we may choose

$$\delta_i \leq \min\{i^{-1}, 2^{-1}i^{-2}[1 + m(\Omega)(\max_{s\in[0,1]} |\partial F(s)| + \max_{s\in[0,1]} |\partial G(s)|)]^{-1}\}. \tag{3.24}$$

For every $i \in \mathbb{N}$ and $\lambda \in [0,\lambda_i]$, let $A_i^\lambda : [0,\infty) \to \mathbb{R}$ be defined as

$$A_i^\lambda(s) = A^\lambda(\tau_{\eta_i}(s)), \tag{3.25}$$

and the energy functional $\mathcal{E}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem $(\mathcal{D}^k_{A_i^\lambda})$ is given by

$$\mathcal{E}_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{k}{2}\int_\Omega u^2 dx - \int_\Omega A_i^\lambda(u(x))dx.$$

Similarly to the proof of Theorem 3.1.1, it can be proved that for every $i \in \mathbb{N}$ and $\lambda \in [0,\lambda_i]$, the function A_i^λ verifies the hypotheses of Theorem 3.2.1. Accordingly, for every $i \in \mathbb{N}$ and $\lambda \in [0,\lambda_i]$:

$$\mathcal{E}_{i,\lambda} \text{ attains its infinum on } W^{\eta_i} \text{ at some } u^0_{i,\lambda} \in W^{\eta_i} \tag{3.26}$$

$$u^0_{i,\lambda}(x) \in [0,\delta_i] \text{ for a.e. } x \in \Omega; \tag{3.27}$$

$$u^0_{i,\lambda} \text{ is a weak solution of } (\mathcal{D}^k_{A_i^\lambda}). \tag{3.28}$$

By the choice of the function A^λ and $k > 0$, $u^0_{i,\lambda}$ is also a solution to the differential inclusion problem $(\mathcal{D}^k_{A^\lambda})$, so $(\mathcal{D}_\lambda)$, thus the claim follows.

(ii) It is clear that for $\lambda = 0$, the set-valued map $\partial A_i^\lambda = \partial A_i^0$ verifies the hypotheses of Theorem 3.3.1. In particular, $\mathcal{E}_i := \mathcal{E}_{i,0}$ is the energy functional associated with problem $(\mathcal{D}^k_{A_i^0})$. Consequently, the elements $u_i^0 := u^0_{i,0}$ verify not only (3.26)-(3.28) but also

$$\mathcal{E}_i(u_i^0) = \min_{W^{\eta_i}} \mathcal{E}_i \leq \mathcal{E}_i(w_{\tilde{s}_i}) < 0 \text{ for all } i \in \mathbb{N}. \tag{3.29}$$

Similarly to Kristály and Moroșanu [30], we may find a $\{\theta_i\}_i$ sequence with negative terms such that $\lim_{i\to\infty} \theta_i = 0$. Due to the expression (3.29) we conclude that

$$\theta_i < \mathcal{E}_i(u_i^0) \leq \mathcal{E}_i(w_{\tilde{s}_i}) < \theta_{i+1}. \tag{3.30}$$

Let us choose

$$\lambda_i^{'} = \frac{\theta_{i+1} - \mathcal{E}_i(w_{\tilde{s}_i})}{m(\Omega)\max_{s\in[0,1]}|G(s)|+1} \text{ and } \lambda_i^{''} = \frac{\mathcal{E}_i(u_i^0) - \theta_i}{m(\Omega)\max_{s\in[0,1]}|G(s)|+1}, \; i \in \mathbb{N},$$

and for a fixed $k \in \mathbb{N}$, set

$$\lambda_k^0 = \min(1, \lambda_1, ..., \lambda_k, \lambda_1{}', ..., \lambda_k{}', \lambda_1{}'', ..., \lambda_k{}'') > 0.$$

Having in our mind these choices, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$ we have

$$\begin{aligned} \mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) &\leq \mathcal{E}_{i,\lambda}(w_{\tilde{s}_i}) = \frac{1}{2}\|w_{\tilde{s}_i}\|_{H_0^1}^2 - \int_\Omega F(w_{\tilde{s}_i}(x))dx - \lambda\int_\Omega G(w_{\tilde{s}_i}(x))dx \\ &= \mathcal{E}_i(w_{\tilde{s}_i}) - \lambda\int_\Omega G(w_{\tilde{s}_i}(x))dx \\ &< \theta_{i+1}, \end{aligned} \tag{3.31}$$

and due to $u_{i,\lambda}^0 \in W^{\eta_i}$ and to the fact that u_i^0 is the minimum point of $\mathcal{E}_i$ on the set W^{η_i}, by the expression (3.30), we also have

$$\mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{E}_i(u_{i,\lambda}^0) - \lambda\int_\Omega G(u_{i,\lambda}^0(x))dx \geq \mathcal{E}_i(u_i^0) - \lambda\int_\Omega G(u_{i,\lambda}^0(x))dx > \theta_i. \tag{3.32}$$

Therefore, by estimations (3.31) and (3.32), for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$, we can find

$$\theta_i < \mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1},$$

thus

$$\mathcal{E}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{E}_{k,\lambda}(u_{k,\lambda}^0) < 0.$$

We observe that $u_i^0 \in W^{\eta_1}$ for every $i \in \{1, ..., k\}$, so $\mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) = \mathcal{E}_{1,\lambda}(u_{i,\lambda}^0)$ because of the truncated function $A_i^\lambda(s)$, see (3.25). Therefore, it follows that for every $\lambda \in [0, \lambda_k^0]$,

$$\mathcal{E}_{1,\lambda}(u_{1,\lambda}^0) < ... < \mathcal{E}_{1,\lambda}(u_{k,\lambda}^0) < 0 = \mathcal{E}_{1,\lambda}(0).$$

Based on these inequalities, it turns out that the elements $u_{1,\lambda}^0, ..., u_{k,\lambda}^0$ are distinct and non-trivial whenever $\lambda \in [0, \lambda_k^0]$.

Now, we are going to prove the estimate (3.1). We have for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$:

$$\mathcal{E}_{1,\lambda}(u_{i,\lambda}^0) = \mathcal{E}_{i,\lambda}(u_{i,\lambda}^0) < \theta_{i+1} < 0.$$

By Lebourg's mean value theorem and the estimation (3.24), we have for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^0]$ that

$$\begin{aligned} \frac{1}{2}\|u_{i,\lambda}^0\|_{H_0^1}^2 &< \int_\Omega F(u_{i,\lambda}^0(x))dx + \lambda \int_\Omega G(u_{i,\lambda}^0(x))dx \\ &\leq m(\Omega)\delta_i[\max_{s\in[0,1]} |\partial F(s)| + \max_{s\in[0,1]} |\partial G(s)|] \\ &\leq \frac{1}{2i^2}. \end{aligned}$$

This completes the proof of Theorem 3.1.2. □

3.4 Proof of Theorems 3.1.3 and 3.1.4

We consider again the differential inclusion problem

$$\begin{cases} -\triangle u(x) + ku(x) \in \partial A(u(x)), \;\; u(x) \geq 0, & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \qquad (\mathcal{D}_A^k)$$

where $k > 0$ and the locally Lipschitz function $A : \mathbb{R}_+ \to \mathbb{R}$ verifies

$(\mathbf{H}_0^\infty)$: $A(0) = 0$;

$(\mathbf{H}_1^\infty)$: $-\infty < \liminf_{s\to\infty} \frac{A(s)}{s^2}$ and $\limsup_{s\to\infty} \frac{A(s)}{s^2} = +\infty$;

$(\mathbf{H}_2^\infty)$: there are two sequences $\{\delta_i\}$, $\{\eta_i\}$ with $0 < \delta_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and

$$\max\{\partial A(s)\} := \max\{\xi : \xi \in \partial A(s)\} \leq 0$$

for every $s \in [\delta_i, \eta_i]$, $i \in \mathbb{N}$.

The counterpart of Theorem 3.3.1 reads as follows.

Theorem 3.4.1. *Let $k > 0$ and assume the hypotheses $(\mathbf{H}_0^\infty)$, $(\mathbf{H}_1^\infty)$ and $(\mathbf{H}_2^\infty)$ hold. Then the differential inclusion problem $(\mathcal{D}_A^k)$ admits a sequence $\{u_i^\infty\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that*

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{3.33}$$

Proof. The proof is similar to the one performed in Theorem 3.3.1; we shall show the differences only. We associate the energy functional $\mathcal{E}_i : H_0^1(\Omega) \to \mathbb{R}$ with problem $(\mathcal{D}_{A_i}^k)$, where $A_i : \mathbb{R} \to \mathbb{R}$ is given by

$$A_i(s) = A(\tau_{\eta_i}(s)), \tag{3.34}$$

with $A(s) = 0$ for $s \leq 0$. One can show that there exists $M_{A_i} > 0$ such that

$$\max|\partial A_i(s)| := \max\{|\xi| : \xi \in \partial A_i(s)\} \leq M_{A_i}$$

for all $s \geq 0$, i.e, hypothesis $(\mathbf{H}_{1,A_i}^{\infty})$ holds. Moreover, $(\mathbf{H}_{2,A_i}^{\infty})$ follows by $(\mathbf{H}_2^{\infty})$. Thus, Theorem 3.3.1 can be applied for all $i \in \mathbb{N}$, i.e., we have an element $u_i^{\infty} \in W^{\eta_i}$ such that

$$u_i^{\infty} \text{ is the minimum point of the functional } \mathcal{T}_i \text{ on } W^{\eta_i}, \tag{3.35}$$

$$u_i^{\infty}(x) \in [0, \delta_i] \text{ for a.e. } x \in \Omega,$$

$$u_i^{\infty} \text{ is a weak solution of } (\mathcal{D}_{A_i}^k).$$

By (3.34), u_i^{∞} turns to be a weak solution also for differential inclusion problem $(\mathcal{D}_A^k)$.

In what follows, we prove that there are infinitely many distinct elements in the sequence $\{u_i^{\infty}\}_i$.

Lemma 3.4.1. *Under the assumptions of the Theorem 3.4.1, we have*

$$\lim_{i\to\infty} \mathcal{E}_i(u_i^{\infty}) = -\infty. \tag{3.36}$$

Proof. By the left part of $(\mathbf{H}_1^{\infty})$ we can find $l_{\infty}^A > 0$ and $\zeta > 0$ such that

$$A(s) \geq -l_{\infty}^A \text{ for all } s > \zeta. \tag{3.37}$$

Let us choose $L_{\infty}^A > 0$ large enough such that

$$\frac{1}{2}C(r,n) + \left(\frac{k}{2} + l_{\infty}^A\right) m(\Omega) < L_{\infty}^A (r/2)^n \omega_n. \tag{3.38}$$

On account of the right part of $(\mathbf{H}_1^{\infty})$, one can fix a sequence $\{\tilde{s}_i\}_i \subset (0, \infty)$ such that $\lim_{i\to\infty} \tilde{s}_i = \infty$ and

$$A(\tilde{s}_i) > L_{\infty}^A \tilde{s}_i^{\,2} \text{ for every } i \in \mathbb{N}. \tag{3.39}$$

We know from $(\mathbf{H}_2^{\infty})$ that $\lim_{i\to\infty} \delta_i = \infty$, therefore one has a subsequence $\{\delta_{m_i}\}_i$ of $\{\delta_i\}_i$ such that $\tilde{s}_i \leq \delta_{m_i}$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$, and recall $w_{si} \in H_0^1(\Omega)$ from (3.5) with $s_i := \tilde{s}_i > 0$. Then $w_{\tilde{s}_i} \in W^{\eta_{m_i}}$ and according to (3.6), (3.37) and (3.39) we have

$$\begin{aligned}
\mathcal{E}_{mi}(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|_{H_0^1}^2 + \frac{k}{2}\int_{\Omega} w_{\tilde{s}_i}^2 dx - \int_{\Omega} A_{m_i}(w_{\tilde{s}_i}(x))dx \\
&= \frac{1}{2}C(r,n)\tilde{s}_i^2 + \frac{k}{2}\int_{\Omega} w_{\tilde{s}_i}^2 dx - \int_{B(x_0,r/2)} A(\tilde{s}_i)dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{w_{\tilde{s}_i}>\zeta\}} A(w_{\tilde{s}_i}(x))dx \\
&\quad - \int_{(B(x_0,r)\setminus B(x_0,r/2))\cap\{w_{\tilde{s}_i}\leq\zeta\}} A(w_{\tilde{s}_i}(x))dx \\
&\leq \left[\frac{1}{2}C(r,n) + \frac{k}{2}m(\Omega) - L_{\infty}^A (r/2)^n \omega_n + l_{\infty}^A m(\Omega)\right] \tilde{s}_i^2 + \tilde{M}_A m(\Omega)\zeta,
\end{aligned}$$

where $\tilde{M}_A = \max\{|A(s)| : s \in [0, \zeta]\}$ does not depend on $i \in \mathbb{N}$. This estimate combined by (3.38) and $\lim_{i\to\infty} \tilde{s}_i = \infty$ yields that

$$\lim_{i\to\infty} \mathcal{E}_{m_i}(w_{\tilde{s}_i}) = -\infty.$$

By equation (3.35), one has

$$\mathcal{E}_{m_i}(u_{m_i}^\infty) = \min_{W^{\eta_{m_i}}} \mathcal{E}_{m_i} \le \mathcal{E}_{m_i}(w_{\tilde{s}_i}),$$

thus our claim holds. □

We notice that the sequence $\{\mathcal{E}_i(u_i^\infty)\}_i$ is non-increasing. Indeed, let $i < k$; due to (3.34) one has that

$$\mathcal{E}_i(u_i^\infty) = \min_{W^{\eta_i}} \mathcal{E}_i = \min_{W^{\eta_i}} \mathcal{E}_k \ge \min_{W^{\eta_k}} \mathcal{E}_k = \mathcal{E}_k(u_k^\infty),$$

which completes the proof of (3.36).

The proof of (3.33) goes in a similar way as in [30]. □

Proof of Theorem 3.1.3. We split the proof into two parts.

(i) *Case* $p = 1$. Let $\lambda \ge 0$ with $\lambda\bar{c} < -l_\infty$ and fix $\tilde{\lambda}_\infty \in \mathbb{R}$ such that $\lambda\bar{c} < \tilde{\lambda}_\infty < -l_\infty$. With these choices, we define

$$k := \tilde{\lambda}_\infty - \lambda\bar{c} > 0 \;\text{ and }\; A(s) := F(s) + \frac{\tilde{\lambda}_\infty}{2}s^2 + \lambda\left(G(s) - \frac{\bar{c}}{2}s^2\right) \;\text{ for every } s \in [0,\infty). \tag{3.40}$$

It is clear that $A(0) = 0$, i.e., $(\mathbf{H}_0^\infty)$ is verified. A similar argument for the p-order perturbation ∂G as before shows that

$$\begin{aligned}\liminf_{s\to\infty} \frac{A(s)}{s^2} &\ge \liminf_{s\to\infty} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda\bar{c}}{2} + \lambda \liminf_{s\to\infty} \frac{G(s)}{s^2} \\ &\ge \liminf_{s\to\infty} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda\bar{c}}{2} + \lambda\underline{c} > -\infty,\end{aligned}$$

and

$$\limsup_{s\to\infty} \frac{A(s)}{s^2} \ge \limsup_{s\to\infty} \frac{F(s)}{s^2} + \frac{\tilde{\lambda}_\infty - \lambda\bar{c}}{2} + \lambda \liminf_{s\to\infty} \frac{G(s)}{s^2} = +\infty,$$

i.e., $(\mathbf{H}_1^\infty)$ is verified.

Since

$$\partial A(s) \subseteq \partial F(s) + \tilde{\lambda}_\infty s + \lambda(\partial G(s) - \bar{c}s), \;\; s \ge 0, \tag{3.41}$$

it turns out that

$$\begin{aligned}\liminf_{s\to\infty}\frac{\max\{\partial A(s)\}}{s} &\leq \liminf_{s\to\infty}\frac{\max\{\partial F(s)\}}{s}+\tilde{\lambda}_\infty-\lambda\overline{c}+\lambda\limsup_{s\to\infty}\frac{\max\{\partial G(s)\}}{s}\\ &= l_\infty+\tilde{\lambda}_\infty\\ &< 0.\end{aligned}$$

By using the upper semicontinuity of $s\mapsto\partial A(s)$, two sequences $\{\delta_i\}_i, \{\eta_i\}_i\subset(0,\infty)$ can be fixed such that $0<\delta_i<s_i<\eta_i<\delta_{i+1}$, $\lim_{i\to\infty}\delta_i=\infty$, and $\max\{\partial A(s)\}\leq 0$ for all $s\in[\delta_i,\eta_i]$ and $i\in\mathbb{N}$. Thus, $(\mathbf{H}_2^\infty)$ is verified as well. By applying the inclusion (3.41) and Theorem 3.3.1 with the choice (3.40), there exists a sequence $\{u_i\}_i\subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases}-\triangle u_i(x)+(\tilde{\lambda}_\infty-\lambda\overline{c})u_i(x)\in\partial F(u_i(x))+\tilde{\lambda}_\infty u_i(x)+\lambda(\partial G(u_i(x))-\overline{c}u_i(x)),\\ x\in\Omega;\\ u_i(x)\geq 0,\ \ x\in\Omega;\\ u_i(x)=0,\ \ x\in\partial\Omega,\end{cases}$$

i.e., u_i solves problem $(\mathcal{D}_\lambda)$, $i\in\mathbb{N}$.

(ii) *Case* $p<1$. Let $\lambda\geq 0$ be arbitrary fixed and choose a number $\lambda_\infty\in(0,-l_\infty)$. Let

$$k:=\lambda_\infty>0\ \text{ and } A(s):=F(s)+\lambda G(s)+\lambda_\infty\frac{s^2}{2}\ \text{for every } s\in[0,\infty). \tag{3.42}$$

Since $F(0)=G(0)=0$, hypothesis $(\mathbf{H}_0^\infty)$ clearly holds. Moreover, by $(\mathbf{G}_1^\infty)$, for sufficiently small $\epsilon>0$ there exists $s_0>0$, such that $(\underline{c}-\epsilon)s^{p+1}\leq G(s)\leq(\overline{c}+\epsilon)s^{p+1}$ for every $s>s_0$. Thus, since $p<1$,

$$\lim_{s\to\infty}\frac{G(s)}{s^2}=\lim_{s\to\infty}\frac{G(s)}{s^{p+1}}s^{p-1}=0.$$

Accordingly, by using (3.42) we obtain that hypothesis $(\mathbf{H}_1^\infty)$ holds. A similar argument as above implies that

$$\liminf_{s\to\infty}\frac{\max\{\partial A(s)\}}{s}\leq l_0+\lambda_\infty<0,$$

and the upper semicontinuity of ∂A implies the existence of two sequences $\{\delta_i\}_i$ and $\{\eta_i\}_i\subset(0,1)$ such that $0<\delta_i<s_i<\eta_i<\delta_{i+1}$, $\lim_{i\to\infty}\delta_i=\infty$, and $\max\{\partial A(s)\}\leq 0$ for all $s\in[\delta_i,\eta_i]$ and $i\in\mathbb{N}$. Therefore, hypothesis $(\mathbf{H}_2^\infty)$ holds. Now, we can apply

Theorem 3.3.1, i.e., there is a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of different elements such that

$$\begin{cases} -\triangle u_i(x) + \lambda_\infty u_i(x) \in \partial A(u_i(x)) \subseteq \partial F(u_i(x)) + \lambda \partial G(u_i(x)) + \lambda_\infty u_i(x), \\ x \in \Omega; \\ u_i(x) \geq 0, \quad x \in \Omega; \\ u_i(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

which means that u_i solves problem $(\mathcal{D}_\lambda)$, $i \in \mathbb{N}$, which completes the proof. □

Proof of Theorem 3.1.4. The proof is done in two steps:

(i) Let $\lambda_\infty \in (0, -l_\infty)$, $\lambda \geq 0$ and define

$$k := \lambda_\infty > 0 \text{ and } A^\lambda(s) := F(s) + \lambda G(s) + \lambda_\infty \frac{s^2}{2} \text{ for every } s \in [0, \infty).$$

One has clearly that $\partial A^\lambda(s) \subseteq \partial F(s) + \lambda_\infty s + \lambda \partial G(s)$ for every $s \in \mathbb{R}$. On account of $(\mathbf{F}_2^\infty)$, there is a sequence $\{s_i\}_i \subset (0, \infty)$ converging to ∞ such that

$$\max\{\partial A^{\lambda=0}(s_i)\} \leq \max\{\partial F(s_i)\} + \lambda_\infty s_i < 0.$$

By the upper semicontinuity of $(s, \lambda) \mapsto \partial A^\lambda(s)$, let the sequences $\{\delta_i\}_i, \{\eta_i\}_i, \{\lambda_i\}_i \subset (0, \infty)$ such that $0 < \delta_i < s_i < \eta_i < \delta_{i+1}$, $\lim_{i\to\infty} \delta_i = \infty$, and

$$\max\{\partial A^\lambda(s)\} \leq 0$$

for all $\lambda \in [0, \lambda_i]$, $s \in [\delta_i, \eta_i]$ and $i \in \mathbb{N}$.

For every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, let $A_i^\lambda : [0, \infty) \to \mathbb{R}$ be defined by

$$A_i^\lambda(s) = A^\lambda(\tau_{\eta_i}(s)), \tag{3.43}$$

and accordingly, the energy functional $\mathcal{E}_{i,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ associated with the differential inclusion problem$(\mathcal{D}^k_{A_i^\lambda})$ is

$$\mathcal{E}_{i,\lambda}(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \frac{k}{2}\int_\Omega u^2 dx - \int_\Omega A_i^\lambda(u(x))dx.$$

Then for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$, the function A_i^λ clearly verifies the hypotheses of Theorem 3.2.1. Accordingly, for every $i \in \mathbb{N}$ and $\lambda \in [0, \lambda_i]$ there exists

$$\mathcal{E}_{i,\lambda} \text{ attains its infimum at some } \tilde{u}^\infty_{i,\lambda} \in W^{\eta_i} \tag{3.44}$$

$$\tilde{u}^\infty_{i,\lambda} \in [0, \delta_i] \text{ for a.e. } x \in \Omega;$$

$$\tilde{u}_{i,\lambda}^{\infty}(x) \text{ is a weak solution of } (\mathcal{D}_{A_i^\lambda}^k). \tag{3.45}$$

Due to (3.43), $\tilde{u}_{i,\lambda}^{\infty}$ is not only a solution to $(\mathcal{D}_{A_i^\lambda}^k)$ but also to the differential inclusion problem $(\mathcal{D}_{A^\lambda}^k)$, so $(\mathcal{D}_\lambda)$.

(ii) For $\lambda = 0$, the function $\partial A_i^\lambda = \partial A_i^0$ verifies the hypotheses of Theorem 3.3.1. Moreover, $\mathcal{E}_i := \mathcal{E}_{i,0}$ is the energy functional associated with problem $(\mathcal{D}_{A_i^0}^k)$. Consequently, the elements $u_i^\infty := u_{i,0}^\infty$ verify not only (3.44)-(3.45) but also

$$\mathcal{E}_{m_i}(u_{m_i}^\infty) = \min_{W^{\eta m_i}} (\mathcal{E}_{m_i}) \le \mathcal{E}_{m_i}(w_{\tilde{s}_i}) \text{ for all } i \in \mathbb{N}, \tag{3.46}$$

where the subsequence $\{u_{m_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ and $w_{\tilde{s}_i} \in W^{\eta_i}$ appear in the proof of Theorem 3.4.1.

Similarly to Kristály and Moroşanu [30], let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty} \theta_i = -\infty$. On account of (3.46) we may assume that

$$\theta_{i+1} < \mathcal{E}_{m_i}(u_{m_i}^\infty) \le \mathcal{E}_{m_i}(w_{\tilde{s}_i}) < \theta_i. \tag{3.47}$$

Let

$$\lambda_i^{'} = \frac{\theta_i - \mathcal{E}_{m_i}(w_{\tilde{s}_i})}{m(\Omega)\max_{s\in[0,1]}|G(s)|+1} \text{ and } \lambda_i^{''} = \frac{\mathcal{E}_{m_i}(u_{m_i}^\infty) - \theta_{i+1}}{m(\Omega)\max_{s\in[0,1]}|G(s)|+1}\ , i \in \mathbb{N},$$

and for a fixed $k \in \mathbb{N}$, we set

$$\lambda_k^\infty = \min(1, \lambda_1, ..., \lambda_k, \lambda_1^{'}, ..., \lambda_k^{'}, \lambda_1^{''}, ..., \lambda_k^{''}) > 0.$$

Then, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^\infty]$, due to (3.47) we have that

$$\begin{aligned}
\mathcal{E}_{m_i,\lambda}(\tilde{u}_{m_i,\lambda}^\infty) &\le \mathcal{E}_{m_i,\lambda}(w_{\tilde{s}_i}) \\
&= \frac{1}{2}\|w_{\tilde{s}_i}\|_{H_0^1}^2 - \int_\Omega F(w_{\tilde{s}_i}(x))dx - \lambda \int_\Omega G(w_{\tilde{s}_i}(x))dx \\
&= \mathcal{E}_{m_i}(w_{\tilde{s}_i}) - \lambda \int_\Omega G(w_{\tilde{s}_i}(x))dx \\
&< \theta_i.
\end{aligned}$$

Similarly, since $\tilde{u}_{m_i,\lambda}^\infty \in W^{\eta m_i}$ and $u_{m_i}^\infty$ is the minimum point of $\mathcal{T}_i$ on the set $W^{\eta m_i}$, on account of (3.47) we have

$$\begin{aligned}
\mathcal{E}_{m_i,\lambda}(\tilde{u}_{m_i,\lambda}^\infty) &= \mathcal{E}_{m_i}(\tilde{u}_{m_i,\lambda}^\infty) - \lambda \int_\Omega G(\tilde{u}_{m_i,\lambda}^\infty)dx \\
&\ge \mathcal{E}_{m_i}(u_{m_i}^\infty) - \lambda \int_\Omega G(\tilde{u}_{m_i,\lambda}^\infty)dx \\
&> \theta_{i+1}.
\end{aligned}$$

Therefore, for every $i \in \{1, ..., k\}$ and $\lambda \in [0, \lambda_k^\infty]$,

$$\theta_{i+1} < \mathcal{E}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) < \theta_i < 0,$$

thus

$$\mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{m_k,\lambda}) < ... < \mathcal{E}_{m_1,\lambda}(\tilde{u}^\infty_{m_1,\lambda}) < 0.$$

Because of (3.43), we notice that $\tilde{u}^\infty_{m_i,\lambda} \in W^{\eta_{m_k}}$ for every $i \in \{1, ..., k\}$, thus $\mathcal{E}_{m_i,\lambda}(\tilde{u}^\infty_{m_i,\lambda}) = \mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{i,\lambda})$. Therefore, for every $\lambda \in [0, \lambda_k^\infty]$,

$$\mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{m_k,\lambda}) < ... < \mathcal{E}_{m_k,\lambda}(\tilde{u}^\infty_{m_1,\lambda}) < 0 = \mathcal{E}_{m_k,\lambda}(0),$$

i.e, the elements $\tilde{u}^\infty_{m_1,\lambda}, ..., \tilde{u}^\infty_{m_k,\lambda}$ are distinct and non-trivial whenever $\lambda \in [0, \lambda_k^\infty]$. The estimate (3.4) follows in a similar manner as in [30]. $\square$

Chapter 4

A non-smooth Neumann problem on compact Riemannian manifolds

In many cases, a recent Ricceri result [44] can be easily invoked to solve partial differential equations involving C^1 functions; for a non-smooth version, see Kristály, Marzantowicz and Varga [25]. Extending their results in several aspects, the aim of this chapter is to present an application of the non-smooth Ricceri's multiplicity theorem [25] to discuss a differential inclusion problem on a compact Riemannian manifolds.

In section 4.1 our main result is established, while section 4.2 stands for its proof. This chapter summerizes results of Szilák [48].

4.1 Main results

Let (M, g) be a connected, compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M. Introducing notations $2^* = \frac{2n}{n-2}$ and $\overline{2}^* = \frac{2(n-1)}{n-2}$, we study the following inhomogeneous Neumann boundary differential inclusion problem

$$\begin{cases} -\Delta_g u(x) + k(x)u \in \lambda K(x)\partial F(u(x)), & x \in M; \\ \frac{\partial u}{\partial \mathbf{n}} \in \mu D(x)\partial G(u(x)), & x \in \partial M, \end{cases} \qquad (\mathcal{D}_{\lambda,\mu})$$

where $k, K : M \to \mathbb{R}$ and $D : \partial M \to \mathbb{R}$ are positive continuous functions, μ and $\lambda > 0$, Δ_g denotes the Laplace-Beltrami operator on (M, g), $\frac{\partial}{\partial \mathbf{n}}$ is the normal derivative with respect to the outward normal ***n*** on ∂M. In addition, F and G are locally Lipschitz functions, ∂F and ∂G denote their generalized gradients in the sense of Clarke and we assume they verify the following conditions:

$(\mathbf{F}_0)$: $F(0) = 0$ and there exists $C_1 > 0$ and $p \in [2, 2^*)$ such that

$$|\xi| \leq C_1(1 + |s|^{p-1}), \forall \xi \in \partial F(s), s \in \mathbb{R};$$

$(\mathbf{F}_1)$: $\limsup_{s\to 0} \frac{\max\{|\xi|:\xi\in\partial F(s)\}}{s} = 0$;

$(\mathbf{F}_2)$: $\limsup_{|s|\to\infty} \frac{F(s)}{s^2} \leq 0$;

$(\mathbf{F}_3)$: there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$;

$(\mathbf{G}_0)$: there exists $C_2 > 0$ and $q \in [2, \overline{2}^*)$ such that

$$|\xi| \leq C_2(1 + |s|^{q-1}), \forall \xi \in \partial G(s), s \in \mathbb{R}.$$

Example 4.1.1. The function $F(s) = \min\{s^3, \sqrt{s}\}$ is locally Lipschitz and one can prove that it satisfies conditions $(\mathbf{F}_0) - (\mathbf{F}_3)$ for $p = 2$.

Remark 4.1.1. Let us note that whenever $K(x)/k(x) = C_3$ and $D(x) = C_4$ for some constants $C_3, C_4 > 0$, furthermore $s \in \mathbb{R}$ solves the inclusion system formed by $s \in C_3\lambda\partial F(s)$ and $0 \in \partial G(s)$ for some $\lambda > 0$, then the constant function $u(x) = s$, $x \in M$, verifies both inclusions in $(\mathcal{D}_{\lambda,\mu})$.

Let us introduce the norm

$$\|u\|_k = \left(\int_M |\nabla u|^2 \mathrm{d}v_g + \int_M k(x)u^2 \mathrm{d}v_g\right)^{\frac{1}{2}}.$$

Let denote $\alpha_m = \min_{x\in M} k(x)$, $\alpha_M = \max_{x\in M} k(x)$ and $K_M = \max_{x\in M} K(x)$. Now we can give the following estimation for $\|\cdot\|_k$:

$$\min\{1, \sqrt{\alpha_m}\}\|u\|_{H^1} \leq \|u\|_k \leq \max\{1, \sqrt{\alpha_M}\}\|u\|_{H^1},$$

which means that $\|\cdot\|_k$ turns out to be equivalent to the $\|\cdot\|_{H^1}$-norm.

The energy functional $\mathcal{E}_{\lambda,\mu} : H^1(M) \to \mathbb{R}$ associated with $(\mathcal{D}_{\lambda,\mu})$ is defined by

$$\mathcal{E}_{\lambda,\mu}(u) = \mathcal{N}(u) + \lambda\mathcal{F}(u) + \mu\mathcal{G}(u),$$

where

$$\mathcal{N} = \frac{1}{2}\|u\|_k^2,$$
$$\mathcal{F}(u) = -\int_M K(x)F(u(x))d\nu_g,$$
$$\mathcal{G}(u) = -\int_{\partial M} D(x)H(u(x))d\nu_g.$$

One can prove that the energy functional is well defined and locally Lipschitz.

We say that $u \in H^1(M)$ is a weak solution of the problem ($\mathcal{D}_{\lambda,\mu}$) if there exists measurable mappings $x \mapsto \xi_x \in \partial F(u(x))$ and $x \mapsto \eta_x \in \partial G(u(x))$ such that for all the test function $w \in H^1(M)$ the functions $x \mapsto \lambda K(x)\xi_x w(x)$ and $x \mapsto \mu D(x)\eta_x w(x)$ belong to $L^1(M)$ and

$$\int_M \nabla_g u(x)\nabla_g w(x)\mathrm{d}v_g + \int_M k(x)u(x)w(x)\mathrm{d}v_g = \lambda \int_M K(x)\xi_x w(x)\mathrm{d}v_g,$$

together with

$$\int_{\partial M} \frac{\partial u}{\partial \mathbf{n}} w(x)\mathrm{d}\sigma_g = \mu \int_{\partial M} D(x)\eta_x w(x)\mathrm{d}\sigma_g,$$

where σ_g stands for the surface measure on ∂M. According to Chang [11], the critical points of the energy functional $\mathcal{E}_{\lambda,\mu}$ are the solutions of our problem ($\mathcal{D}_{\lambda,\mu}$); see also Kristály, Marzantowicz and Varga [25].

We present the main result of this chapter:

Theorem 4.1.1. (Sziłák [48]) *Let $F : M \to \mathbb{R}$ and $G : M \to R$ be functions that fulfill the assumptions $(\mathbf{F}_0) - (\mathbf{F}_3)$ and $(\mathbf{H}_0)$, respectively. Then there exist a number η and a non-degenerate compact interval $A \subset (0, +\infty)$ such that for every $\lambda \in A$ there exists $\mu_0 \in (0, \lambda + 1]$ so that whenever μ is small enough i.e. $\mu \in [0, \mu_0]$, the inclusion $(\mathcal{D}_{\lambda,\mu})$ has at least three solutions which are in norm less than η.*

The proof of Theorem 4.1.1 uses the non-smooth variational calculus recalled in subsection 2.4.1 together with Theorem 2.4.3; the main ingredients are the three critical points theorem of Ricceri for locally Lipschitz functions and the non-smooth Palais-Smale condition.

4.2 Proof of Theorem 4.1.1

We assume the assumptions of Theorem 4.1.1 are fulfilled. First we need a lemma, whose proof requires the function

$$\beta(t) = \inf\{\mathcal{F}(u) : u \in H^1(M), \mathcal{N}(u) < t\}.$$

Lemma 4.2.1. *We have that*

$$\lim_{t \to 0+} \frac{\beta(t)}{t} = 0. \tag{4.1}$$

Proof. The proof is similar to that in Kristály, Marzantowicz and Varga [25]. Combining Lebourg's Mean-Value Theorem with assumptions ($\mathbf{F}_0$) and ($\mathbf{F}_1$), for any $\varepsilon > 0$ one has $L_\varepsilon > 0$ such that

$$|F(t)| \leq C_1 \varepsilon t^2 + C_1 L_\varepsilon |t|^p \text{ for all } t \in \mathbb{R}.$$

Thus, we obtain that

$$\mathcal{F}(u) \geq -C_1 \varepsilon K_M \|u\|_{H^1}^2 - C_1 K_M L_\varepsilon K_p^p \|u\|_{H^1}^p, \tag{4.2}$$

where $K_p > 0$ denotes the constant in the continuous Sobolev embedding $H^1(M) \hookrightarrow L^p(M)$.

Let us define a set for $t > 0$ by

$$S = \{u \in H^1(M) : \|u\|_{H^1}^2 < 2t\}.$$

Let $t > 0$. We recall the fact that $\|\cdot\|_k$ is equivalent to $\|\cdot\|_{H^1}$-norm, consequently applying (4.2) it turns out that

$$0 \geq \frac{\inf_{u \in S} \mathcal{F}(u)}{t} \geq -C_1 2\varepsilon K_M - C_1 K_M L_\varepsilon K_p^p 2^{\frac{p}{2}} t^{\frac{p}{2}-1},$$

which clearly proves Lemma 4.2.1 due to arbitrariness of ε and by the fact that $t \to 0+$. $\square$

Recalling ($\mathbf{F}_3$), let us define a function $u_0 \in H^1(M)$ by $u_0 := s_0$ for all $x \in M$. Thus, if we combine the fact that $F(0) = 0$ with $F(s_0) > 0$, $s_0 \neq 0$, gives the estimate $\mathcal{F}(u_0) < 0$. Accordingly, applying (4.1) one can fix $t_0 \in (0, \mathcal{N}(u_0))$ and $\rho_0 > 0$ such that

$$-\beta(t_0) < \rho_0 < -t_0 \frac{\mathcal{F}(u_0)}{\mathcal{N}(u_0)} < -\mathcal{F}(u_0). \tag{4.3}$$

At this point we define the function $\gamma : H^1(M) \times I \to \mathbb{R}$ by

$$\gamma(u, \lambda) = \mathcal{N}(u) + \lambda \mathcal{F} + \lambda \rho_0, \text{ where } I = [0, \infty).$$

Similarly to Kristály, Marzantowicz and Varga [25], applying (4.3), the following lemma can be easily proved.

Lemma 4.2.2. *We have that*

$$\sup_{\lambda \in I} \inf_{u \in H^1(M)} \gamma(u, \lambda) < \inf_{u \in H^1(M)} \sup_{\lambda \in I} \gamma(u, \lambda).$$

In what follows, fixing a linear function around the origin $g \in \mathcal{G}_\tau$, $\tau \geq 0$ (see [25]), we study the analytic properties of the modified energy functional $\tilde{\mathcal{E}}_{\mu,\lambda} : H^1(M) \to \mathbb{R}$, namely

$$\tilde{\mathcal{E}}_{\mu,\lambda}(u) = \frac{1}{2}\|u\|_k^2 + \lambda \mathcal{F}(u) + \mu (g \circ \mathcal{G})(u).$$

4.2.1 Analytic characteristics of $\tilde{\mathcal{E}}_{\mu,\lambda}$

We shall prove that the modified energy functional $\tilde{\mathcal{E}}_{\mu,\lambda}$ is coercive, bounded from below and satisfies the Palais-Smale compactness condition.

It turns out that $\tilde{\mathcal{E}}_{\mu,\lambda}$ is locally Lipschitz and well-defined, furthermore applying $(\mathbf{F}_2)$ one can prove that $\frac{1}{2}\|\cdot\|_k^2 + \lambda\mathcal{F}$ is coercive. Consequently it follows that $\tilde{\mathcal{E}}_{\mu,\lambda}$ is coercive as well.

Lemma 4.2.3. *The locally Lipschitz functional $\tilde{\mathcal{E}}_{\mu,\lambda}$ satisfies* $(\mathrm{PS})_c$ *condition at every level* $c \in \mathbb{R}$.

Proof. Let a series $\{u_k\}_k$ be a Palais-Smale sequence for $\tilde{\mathcal{E}}_{\mu,\lambda}$, i.e. $m(u_k) \to 0$ as $k \to 0$, where $m(u_k) = min\{\|\xi\|_* : \xi \in \partial\tilde{\mathcal{E}}_{\mu,\lambda}(u_k)\}$. Since the modified energy functional is coercive, the sequence $\{u_k\}$ is bounded on $H^1(M)$. Thus, up to a subsequent, $\{u_k\}_k$ converges weakly in $H^1(M)$ and strongly in $L^p(M)$, $p \in [2, 2^*)$ and in $L^q(\partial M)$, $q \in [2, \overline{2}^*)$. Our objective is to show that for up to a subsequence, $\{u_k\}_k$ converges strongly in $H^1(M)$.

The generalized directional derivative of the modified energy functional for all $u, v \in H^1(M)$ is given by

$$(\tilde{\mathcal{E}}_{\mu,\lambda})^\circ(u, v) \leq \frac{1}{2}\langle\mathcal{N}'(u); v\rangle + \lambda\mathcal{F}^\circ(u, v) + \mu(g \circ \mathcal{G})^\circ(u, v).$$

Calling latter inequality with parameters $u = u_k$ and $v = u - u_k$, then $u = u$ and $v = u_k - u$ one has

$$\begin{aligned}\mathcal{N}(u_k - u) \leq\ & \lambda\big(\mathcal{F}^\circ(u_k, u - u_k) + \mathcal{F}^\circ(u, u_k - u)\big)\\ & + \mu\big((g \circ \mathcal{G})^\circ(u_k, u - u_k) + (g \circ \mathcal{G})^\circ(u, u_k - u)\big)\\ & - \tilde{\mathcal{E}}^\circ_{\mu,\lambda}(u_k, u - u_k) - \tilde{\mathcal{E}}^\circ_{\mu,\lambda}(u, u_k - u).\end{aligned}$$

On one hand, since $m(u_k) \to 0$, it follows that

$$\liminf_{k\to\infty}\big(\tilde{\mathcal{E}}^\circ_{\mu,\lambda}(u, u_k - u) + \tilde{\mathcal{E}}^\circ_{\mu,\lambda}(u_k, u - u_k)\big) \geq 0.$$

On the other hand, combining the fact that $u_k \to u$ in $L^p(M)$ and in $L^q(\partial M)$ with the upper semicontinuity of $\mathcal{F}^\circ(\cdot,\cdot)$ imply that $\limsup_{k\to\infty} \mathcal{F}^\circ(u_k, u - u_k) \leq \mathcal{F}^\circ(u, 0) = 0$. Similarly, $\limsup_{k\to\infty} \mathcal{G}^\circ(u_k, u - u_k) \leq 0$ follows.

Combining estimations above with the fact that $\mathcal{N}(u_k - u) \geq 0$ we obtain that $u \to u_k$ strongly in $H^1(M)$, which imply that the $(\mathrm{PS})_c$ condition follows. $\square$

4.3 Three critical points

At this point we can apply the non-smooth critical points theorem, see Theorem 2.4.3, by choosing $X = H^1(M)$, $\tilde{X}_1 = L^p(M)$, $\tilde{X}_2 = L^q(\partial M)$ with $p \in [2, 2^*)$, $q \in [2, \overline{2}^*)$, $\Lambda = I = [0, +\infty)$, $h(t) = \frac{t^2}{2}$, $t \geq 0$ and let us fix $g \in \mathcal{G}_\tau$, $\tau \geq 0$, $\lambda \in \Lambda$ and $\mu \in [0, \lambda + 1]$. Thus, since $\frac{1}{2}\| \cdot \|^2 + \lambda\mathcal{F}$ is coercive on $H^1(M)$ for all $\lambda \in I$, the Lemma 4.2.2 holds and the functional $\tilde{\mathcal{E}}_{\mu,\lambda}(u) = \frac{1}{2}\|u\|_k^2 + \lambda\mathcal{F}(u) + \mu(g \circ \mathcal{G})(u)$ for all $u \in H^1(M)$ fulfills the (PS)$_c$ condition, we have at least three critical points with $H^1(M)$ norm less than η and Theorem 4.1.1 follows. $\square$

Remark 4.3.1. Differential inclusions of the above type can be investigated also in the setting of Finsler manifolds, see [38]; for unity of the exposition, we do not enter into details.

Part II

Differential inclusions - non-compact case

Chapter 5

Elliptic differential inclusions on non-compact Riemannian manifolds

PDEs may appear not only on bounded domains of Euclidean structures; physical and mechanical phenomena quite frequently require the application of inclusion problems on the broad class of curved spaces. Considering a complete, n-dimensional, non-compact Riemannian manifold (M, g) with certain curvature restrictions ($n \geq 3$), we study the following differential inclusion problem

$$\mathcal{L}u(x) = -\Delta_g u(x) - \mu \frac{u(x)}{d_g^2(x_0, x)} + u(x) \in \lambda\alpha(x)\partial F(u(x)), \;\; x \in M. \qquad (\mathcal{D})$$

Here $\mathcal{L}$ denotes an elliptic type operator, Δ_g represents the Laplace-Beltrami operator on (M, g), $d_g : M \times M \to \mathbb{R}$ is the distance function associated with the Riemannian metric g, $x_0 \in M$ is a fixed point, $\mu, \lambda \in \mathbb{R}$ are some parameters. The function $\alpha : M \to \mathbb{R}$ is a measurable potential, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and ∂F stands for the Clarke subdifferential of F.

On one hand, variational elliptic differential inclusions as $(\mathcal{D})$ – or slightly different versions of them formulated in terms of variational-hemivariational inequalities – have been subject of several investigation in the last three decades, mostly in Euclidean spaces (both for bounded and unbounded domains), see e.g. Kristály and Varga [33], Liu, Liu and Motreanu [35], Liu, Livrea, Motreanu and Zeng [36], etc. On the other hand, various forms of $(\mathcal{D})$ have been investigated both on compact and non-compact Riemannian manifolds (mostly without the singular term), see e.g. Berchio, Ferrero and Grillo [7], Bonanno, Molica Bisci and Rădulescu [8], etc.

We consider problem $(\mathcal{D})$ under two different curvature conditions. More precisely, we assume that the Riemannian manifold (M, g) satisfies one of the following conditions:

(i) Cartan-Hadamard manifold,

(ii) The Ricci curvature is non-negative.

This chapter is devoted to focus on non-existence, existence and multiplicity results for the differential inclusion problem $(\mathcal{D})$ by assuming curvature hypothesis (i) or (ii), together with additional grows conditions on the locally Lipschitz function F (at the origin and infinity). It turns out that the variational methods cannot be used directly. Indeed, since such manifolds are not compact, it is not possible to use certain Sobolev embeddings; as we mentioned in the introduction, the lack of compactness has to be compensated with the application of isometric actions and the principle of symmetric criticality.

The chapter is organized as follows. In section 5.2 we prove a non-existence result, established within Theorem 5.1.1. In section 5.3 we discuss our first existence/multiplicity results in the sub-quadratic case, by proving Theorem 5.1.2. Finally, section 5.4 is devoted to handle the super-quadratic case, i.e., Theorem 5.1.3.

This chapter summerize results of Kristály, Mezei, and Szilák [28].

5.1 Main theorems

First, we discuss non-existence results under the above special curvature conditions; to do this, we assume on the potential $\alpha : M \to \mathbb{R}$ that

$(\mathbf{H}_\alpha)$: $\alpha \geq 0$ and $\alpha \in L^1(M) \cap L^\infty(M) \setminus \{0\}$,

and additionally on the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ that

$(\mathbf{H}_0)$: there exists $C_0 > 0$ such that

$$|\xi| \leq C_0|t|, \ \ \forall \xi \in \partial F(t), \ t \in \mathbb{R}.$$

In the sequel we need the definition of the weak solution associated to the problem $(\mathcal{D})$: an element $u \in H^1(M)$ is a weak solution of $(\mathcal{D})$ if there exists a measurable selection $x \mapsto \xi_x \in \partial F(u(x))$ such that the map $x \mapsto \alpha(x)\xi_x w(x)$ belongs to $L^1(M)$ for every test-function $w \in H^1(M)$ and one has

$$\int_M \nabla_g u(x) \nabla_g w(x) \mathrm{d}v_g - \mu \int_M \frac{u(x)w(x)}{d_g^2(x_0,x)} \mathrm{d}v_g + \int_M u(x)w(x)\mathrm{d}v_g \tag{5.1}$$
$$= \lambda \int_M \alpha(x)\xi_x w(x) \mathrm{d}v_g.$$

The first result of the present chapter reads as follows.

Theorem 5.1.1. (Kristály, Mezei, and Szilák [28]) (Non-existence) *Let (M,g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and assume that the potential $\alpha : M \to \mathbb{R}$ and the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfy assumptions $(\mathbf{H}_\alpha)$ and $(\mathbf{H}_0)$, respectively. Assume in addition that one of the following curvature conditions holds*:

(i) $\mathbf{K} \leq -\kappa$ *for some* $\kappa \geq 0$, (M,g) *is simply connected and*

(i1) either $\kappa = 0$, $\mu \leq \frac{(n-2)^2}{4}$ *and* $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$,

(i2) or $\kappa > 0$, $\mu \leq \frac{(n-2)^2}{4}$ *and* $(n-2)^2(|\lambda| C_0 \|\alpha\|_{L^\infty} - 1) \leq (n-1)^2 \left(\frac{(n-2)^2}{4} - \mu_+\right)\kappa$, *where* $\mu_+ = \max(\mu, 0)$;

(ii) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mu \leq \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$ *and* $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$.

Then the differential inclusion $(\mathcal{D})$ has only the zero solution.

The proof of Theorem 5.1.1 is based on a direct computation combined with Hardy-type inequalities and sharp spectral gap estimates on Riemannian manifolds.

In order to produce existence or even multiplicity of non-zero solutions to $(\mathcal{D})$, we require on the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ the following assumptions:

$(\mathbf{H})_1$: $\lim\limits_{t \to 0} \dfrac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0$;

$(\mathbf{H})_2$: $\lim\limits_{|t| \to \infty} \dfrac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0$;

$(\mathbf{H})_3$: $F(0) = 0$ and there exist $t_0^- < 0 < t_0^+$ such that $F(t_0^\pm) > 0$.

Remark 5.1.1. Note that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ mean that the function $t \mapsto \max\{|\xi| : \xi \in \partial F(t)\}$ is *superlinear at the origin* and *sublinear at infinity*, respectively; in particular, by using Lebourg's mean value theorem, we observe that F is *sub-quadratic at infinity*.

Remark 5.1.2. By the upper semicontinuity of the set-valued function $t \mapsto \partial F(t)$ and conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we can observe that the hypothesis $(\mathbf{H}_0)$ is also valid for a suitably large value of $C_0 > 0$; in particular, Theorem 5.1.1 can be applied (under the assumptions $(\mathbf{H}_\lambda)$, $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$), and for sufficiently 'small' values of $|\lambda|$ only the zero solution exists for the differential inclusion $(\mathcal{D})$.

Whenever λ is large enough, multiplicity result can be established involving additional assumptions in order to balance the lack of compactness of the Riemannian manifolds we are dealing with. The next theorem provides a multiplicity result with a sub-quadratic nonlinearity at the infinity.

Theorem 5.1.2. (Kristály, Mezei, and Szilák [28]) (Multiplicity: sub-quadratic nonlinearity at infinity) *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and G be a compact connected subgroup of $\mathsf{Isom}_g(M)$ such that $\mathsf{Fix}_M(G) = \{x_0\}$ for the same $x_0 \in M$ as in problem $(\mathcal{D})$. Let $\alpha : M \to \mathbb{R}$ be a potential satisfying $(\mathbf{H}_\alpha)$ which depends only on $d_g(x_0, \cdot)$ and the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfying assumptions $(\mathbf{H}_i)$, $i \in \{1, 2, 3\}$, respectively. In addition, we assume that one of the following curvature assumptions holds*:

- *(i)* (M, g) *is of Cartan-Hadamard-type and*
- *(ii)* $\mathsf{Ric}_{(M,g)} \geq 0$, $\mathsf{AVR}_{(M,g)} > 0$, $0 \leq \mu < \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$ *and G is coercive.*

Then there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ the differential inclusion $(\mathcal{D})$ has at least four non-zero G-invariant solutions in $H^1(M)$.

The proof of Theorem 5.1.2 uses elements from the variational calculus described in section 2.4.1. In the sequel, we establish a counterpart of Theorem 5.1.2 whenever F is *super-quadratic at infinity*. In order to prove more existence and multiplicity results, we introduce additional constraints on the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$:

$(\mathbf{H}_4)$: $F(0) = 0$ and there exist $\nu > 2$ and $C > 0$ such that

$$2F(t) + F^0(t; -t) \leq -C|t|^\nu, \ \ \forall t \in \mathbb{R};$$

$(\mathbf{H}_5)$: there is $q \in \left(2, 2 + \frac{4}{n}\right)$ such that $\max\{|\xi| : \xi \in \partial F(t)\} = O(|t|^{q-1})$ as $|t| \to \infty$. Here, $F^0(t; s)$ is the generalized directional derivative of F at the point $t \in \mathbb{R}$ and direction $s \in \mathbb{R}$. Note that by $(\mathbf{H}_1)$ and $(\mathbf{H}_4)$, F is super-quadratic at infinity.

Theorem 5.1.3. (Kristály, Mezei, and Szilák [28]) (Existence/Multiplicity: super-quadratic nonlinearity at infinity) *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and G be a compact connected subgroup of* $\mathsf{Isom}_g(M)$ *such that*

$\mathsf{Fix}_M(G) = \{x_0\}$ *for the same* $x_0 \in M$ *as in problem* $(\mathcal{D})$*. Let* $\alpha \in L^\infty(M)$ *be a potential which depends only on* $d_g(x_0, \cdot)$ *and* $\mathrm{essinf}_{x\in M}\alpha(x) = \alpha_0 > 0$*, while the locally Lipschitz function* $F : \mathbb{R} \to \mathbb{R}$ *satisfies the assumptions* $(\mathbf{H}_i)$*,* $i \in \{1, 4, 5\}$*, respectively. If one of the curvature assumptions* (i) *or* (ii) *holds from Theorem 5.1.2, then for every* $\lambda > 0$ *the differential inclusion* $(\mathcal{D})$ *has at least a non-zero G-invariant solution in* $H^1(M)$*. In addition, if* F *is an even function,* $(\mathcal{D})$ *has infinitely many distinct G-invariant solutions in* $H^1(M)$*.*

The proof is based on the same geometric arguments as in Theorem 5.1.2 (curvature constraints, isometric actions), combined with the non-smooth mountain pass or fountain theorem involving the Cerami compactness condition.

5.2 Non-existence of solutions: proof of Theorem 5.1.1

Let $u \in H^1(M)$ be a solution of $(\mathcal{D})$, i.e., relation (5.1) holds for every $w \in H^1(M)$. Let us choose $w = u$ in (5.1), we obtain

$$\int_M |\nabla_g u(x)|^2 \mathrm{d}v_g - \mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g + \int_M u^2(x)\mathrm{d}v_g = \lambda \int_M \alpha(x)\xi_x u(x)\mathrm{d}v_g,$$

where $\xi_x \in \partial F(u(x))$ is a suitable selection, $x \in M$, such that $x \mapsto \alpha(x)\xi_x v(x)$ belongs to $L^1(M)$. Combining assumptions $(\mathbf{H}_\alpha)$ and $(\mathbf{H}_0)$ with the latter relation we establish the estimation

$$\begin{aligned} \int_M |\nabla_g u(x)|^2 \mathrm{d}v_g - \mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g + \int_M u^2(x)\mathrm{d}v_g & \\ \leq |\lambda| C_0 \|\alpha\|_{L^\infty} \int_M u^2(x)\mathrm{d}v_g. & \end{aligned} \tag{5.2}$$

Assume by contradiction that $u \neq 0$.

5.2.1 Proof of (i): $\mathbf{K} \leq -\kappa$ for some $\kappa \geq 0$

We distinguish two cases.

(i1) Let $\kappa = 0$. If $\mu \leq \frac{(n-2)^2}{4}$, by the Hardy inequality (2.6) and relation (5.2), it turns out that

$$\int_M u^2(x)\mathrm{d}v_g < |\lambda| C_0 \|\alpha\|_{L^\infty} \int_M u^2(x)\mathrm{d}v_g;$$

here we used the fact that equality cannot occur in the Hardy inequality (2.6) unless $u = 0$. Consequently, if $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$, we arrive to a contradiction, i.e., we necessarily have $u = 0$, concluding the proof of (i1).

(i2) Let $\kappa > 0$. Assume first that $0 < \mu \leq \frac{(n-2)^2}{4}$. Then by the Hardy inequality (2.6) we have that

$$\mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g < \frac{4\mu}{(n-2)^2} \int_M |\nabla_g u(x)|^2 \mathrm{d}v_g,$$

where we again used the fact that no equality occurs in (2.6) for non-zero functions. Thus, by (5.2) it follows that

$$\left(1 - \frac{4\mu}{(n-2)^2}\right) \int_M |\nabla_g u(x)|^2 \mathrm{d}v_g < (|\lambda| C_0 \|\alpha\|_{L^\infty} - 1) \int_M u^2(x) \mathrm{d}v_g. \tag{5.3}$$

First, if $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$, since $\mu \leq \frac{(n-2)^2}{4}$, latter relation gives a contradiction. Second, if $|\lambda| C_0 \|\alpha\|_{L^\infty} > 1$, by our assumption $(n-2)^2(|\lambda| C_0 \|\alpha\|_{L^\infty} - 1) \leq (n-1)^2 \left(\frac{(n-2)^2}{4} - \mu\right) \kappa$ we obtain that $\mu < \frac{(n-2)^2}{4}$; moreover, relation (5.3) and the assumption imply that

$$\int_M |\nabla_g u(x)|^2 \mathrm{d}v_g < \frac{(n-1)^2}{4} \kappa \int_M u^2(x) \mathrm{d}v_g.$$

However, the latter inequality is in contradiction to McKean's spectral gap theorem, see (2.7). Therefore, we necessarily have $u = 0$, concluding the proof of (i2) for $\mu > 0$.

If $\mu \leq 0$, then our assumption reduces to $|\lambda| C_0 \|\alpha\|_{L^\infty} - 1 \leq \frac{(n-1)^2}{4} \kappa$ and by (5.2) one has that

$$\int_M |\nabla_g u(x)|^2 \mathrm{d}v_g \leq (|\lambda| C_0 \|\alpha\|_{L^\infty} - 1) \int_M u^2(x) \mathrm{d}v_g.$$

Therefore, we obtain that

$$\int_M |\nabla_g u(x)|^2 \mathrm{d}v_g \leq \frac{(n-1)^2}{4} \kappa \int_M u^2(x) \mathrm{d}v_g.$$

Since no equality occurs in McKean's spectral gap estimate (2.7) for any non-zero function $u \in H^1(M)$, we arrive to a contradiction. In conclusion, we necessarily have that $u = 0$, which ends the proof of (i2) also for $\mu \leq 0$.

5.2.2 Proof of (ii): $\mathbf{Ric}_{(M,g)} \geq 0$

Since $\mu \leq \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$, the Hardy inequality from (2.9) (together with the fact that no non-zero function realizes the equality) and relation (5.2) imply that

$$\int_M u^2(x) \mathrm{d}v_g < |\lambda| C_0 \|\alpha\|_{L^\infty} \int_M u^2(x) \mathrm{d}v_g.$$

Consequently, if $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$, we arrive to a contradiction; thus $u = 0$. This ends the proof of (ii). $\square$

5.3 Sub-quadratic case: proof of Theorem 5.1.2

Throughout this section we assume the hypotheses in Theorem 5.1.2 are satisfied. The proof is divided into several steps.

5.3.1 Truncation and non-smooth energy functional (Step 1)

Let $s^+ = \max(s, 0)$ be the non-negative part of s. Introducing the truncated locally Lipschitz function $F^+(s) = F(s^+)$, $s \in \mathbb{R}$, the energy functional $\mathcal{E}^+ : H^1(M) \to \mathbb{R}$ to the slightly modified problem $(\mathcal{D})$, considering F^+ instead of F, is defined as

$$\mathcal{E}^+(u) = \frac{1}{2}\mathcal{N}_\mu(u) - \lambda \mathcal{F}^+(u),$$

where

$$\mathcal{N}_\mu(u) = \int_M |\nabla_g u(x)|^2 \mathrm{d}v_g - \mu \int_M \frac{u^2(x)}{d_g^2(x_0, x)} \mathrm{d}v_g + \int_M u^2(x) \mathrm{d}v_g$$

and

$$\mathcal{F}^+(u) = \int_M \alpha(x) F^+(u(x)) \mathrm{d}v_g.$$

On one hand, it is clear that $\mathcal{N}_\mu$ is of class C^1 on $H^1(M)$ and due to the Hardy inequalities (i.e., (2.6) and (2.9)), for the corresponding values of μ from the statement of the theorem, $\mathcal{N}_\mu^{1/2}$ turns out to be equivalent to the usual norm $\|\cdot\|_{H^1}$ on $H^1(M)$, i.e.,

$$c_\mu \|u\|_{H^1}^2 \leq \mathcal{N}_\mu(u) \leq \|u\|_{H^1}^2, \ \ \forall u \in H^1(M), \tag{5.4}$$

where

$$0 < c_\mu = \begin{cases} 1 - \frac{4\mu}{(n-2)^2}, & \text{in the case (i);} \\ 1 - \mathsf{AVR}_{(M,g)}^{-\frac{2}{n}} \frac{4\mu}{(n-2)^2} & \text{in the case (ii).} \end{cases}$$

Lemma 5.3.1. *The truncated function $\mathcal{F}^+$ is well-defined and locally Lipschitz on $H^1(M)$.*

Proof. The fact that $\mathcal{F}^+$ is well-defined follows implicitly by the following argument. By $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, for every $\epsilon > 0$ there exists $\delta_\epsilon \in (0, 1)$ such that

$$|\xi| \leq \epsilon t, \ \forall \xi \in \partial F^+(t), \ \forall 0 < t < \delta_\epsilon \ \&\ t > \delta_\epsilon^{-1}. \tag{5.5}$$

Fix $\epsilon_0 > 0$. Since ∂F^+ is an upper semicontinuous set-valued map with non-empty compact values, we also have for some $K_{\epsilon_0} > 0$ that $|\xi| \leq K_{\epsilon_0} t$ for every $\xi \in \partial F^+(t)$ and $t \in [\delta_{\epsilon_0}, \delta_{\epsilon_0}^{-1}]$. The latter fact with (5.5) implies that

$$|\xi| \leq C_{\epsilon_0} t, \ \forall \xi \in \partial F^+(t), \ \forall t > 0,$$

where $C_{\epsilon_0} = \max\{\epsilon_0, K_{\epsilon_0}\}$. Now, let $u \in H^1(M)$ and U_u be any open bounded neighborhood of u in $H^1(M)$, i.e., for some $K > 0$ we have $\|w\|_{H^1} \leq K$ for every $w \in U_u$. If $u_1, u_2 \in U_u$, then by Lebourg's mean value theorem, for a.e. $x \in M$ there exist $\gamma \in [0,1]$ and $\xi_x^\gamma \in \partial F^+((1-\gamma)u_1(x) + \gamma u_2(x))$ such that

$$\begin{aligned} |F^+(u_1(x)) - F^+(u_2(x))| &= |\xi_x^\gamma||u_1(x) - u_2(x)| \\ &\leq C_{\epsilon_0}(|u_1(x)| + |u_2(x)|)|u_1(x) - u_2(x)|. \end{aligned}$$

By Hölder's inequality and the trivial embedding $H^1(M) \subset L^2(M)$, we have that

$$\begin{aligned} |\mathcal{F}^+(u_1) - \mathcal{F}^+(u_2)| &\leq \int_M \alpha(x)|F^+(u_1(x)) - F^+(u_2(x))| \mathrm{d}v_g \\ &\leq 2C_{\epsilon_0}\|\alpha\|_{L^\infty} K \|u_1 - u_2\|_{H^1}, \end{aligned}$$

which means that $\mathcal{F}^+$ is Lipschitz on $H^1(M)$. □

On one hand, applying argument as in Clarke [12, Section 2.7] (see also Costea, Kristály and Varga [13]) shows that for every closed subspace W of $H^1(M)$ we have that

$$\partial(\mathcal{F}^+|_W)(u) \subseteq \int_M \alpha(x)\partial F^+(u(x))\mathrm{d}v_g, \ \ \forall u \in W;$$

here, $\mathcal{F}^+|_W$ is the restriction of the functional $\mathcal{F}^+$ to the subspace W and the latter inclusion has the following interpretation: to every $\xi \in \partial(\mathcal{F}^+|_W)(u)$ there exists a measurable selection $x \mapsto \xi_x \in \partial F^+(u(x))$ such that the map $x \mapsto \alpha(x)\xi_x w(x)$ belongs to $L^1(M)$ for every $w \in W$ and

$$\langle \xi, w \rangle = \int_M \alpha(x)\xi_x w(x)\mathrm{d}v_g.$$

On the other hand, by using Fatou's lemma, Lebourg's mean value theorem, Lebesgue's dominated convergence theorem, and a careful limiting argument, see e.g. Kristály [23] in the Euclidean setting, it turns out that

$$(\mathcal{F}^+|_W)^0(u; w) \leq \int_M \alpha(x)(F^+)^0(u(x); w(x))\mathrm{d}v_g, \ \ \forall u, w \in W. \tag{5.6}$$

Let $u \in H^1(M)$ be a critical point of $\mathcal{E}^+$, i.e., $0 \in \partial\mathcal{E}^+(u)$. We are going to prove that u is a non-negative solution to the differential inclusion $(\mathcal{D})$. First we have that

$$\frac{1}{2}\mathcal{N}_\mu'(u) \in \lambda\partial\mathcal{F}^+(u),$$

i.e., for every test-function $w \in H^1(M)$ one has

$$\int_M \nabla_g u(x)\nabla_g w(x)\mathrm{d}v_g - \mu\int_M \frac{u(x)w(x)}{d_g^2(x_0,x)}\mathrm{d}v_g + \int_M u(x)w(x)\mathrm{d}v_g$$
$$= \lambda\int_M \alpha(x)\xi_x w(x)\mathrm{d}v_g,$$

with the above interpretation for the right hand side.

Let $u_- = \min(0,u)$ be the non-positive part of u and note that it belongs to the space $H^1(M)$, see Hebey [19, Proposition 2.5]. If we put $v = u_-$ into the latter relation, we obtain that $\xi_x u_-(x) = 0$ for a.e. $x \in M$ since $\xi_x \in \partial F^+(u(x))$ (thus $\xi_x = 0$ whenever $u(x) < 0$). In consequence, $\mathcal{N}_\mu(u_-) = 0$, thus $u_- = 0$, i.e., $u \geq 0$. In particular, $\xi_x \in \partial F^+(u(x)) = \partial F(u(x))$, therefore the latter relation is precisely (5.1), which means that $u \in H^1(M)$ is a non-negative solution of $(\mathcal{D})$.

5.3.2 Isometry actions (Step 2)

Recalling the notions from subsection 2.4.2, we prove the following lemma.

Lemma 5.3.2. *The locally Lypschitz energy functional $\mathcal{E}^+$ is G-invariant.*

Proof. (i): Since G contains isometries of (M,g), the functionals $u \mapsto \int_M |\nabla_g u(x)|^2\mathrm{d}v_g$ and $u \mapsto \int_M u^2(x)\mathrm{d}v_g$ are both G-invariant; in particular, $\|\sigma u\|_{H^1} = \|u\|_{H^1}$ for every $\sigma \in G$ and $u \in H^1(M)$. Indeed,

$$\begin{aligned}
\|\sigma u\|_{H^1}^2 &= \int_M (|\sigma u(x))|^2 + |\nabla_g(\sigma u(x))|^2\mathrm{d}v_g(x) \\
&= \int_M |u(\sigma^{-1}(x))|^2 + |\nabla_g(\sigma^{-1}(x))|^2\mathrm{d}v_g(x) \\
&= \int_M |u(y)|^2 + |\nabla_g u(y)|^2\mathrm{d}v_g(y) \\
&= \|u\|_{H^1}^2.
\end{aligned} \tag{5.7}$$

(ii) Since $\mathsf{Fix}_M(G) = \{x_0\}$, it turns out that for every $\sigma \in G$ and $y \in M$, we have $d_g(x_0,\sigma(y)) = d_g(\sigma(x_0),\sigma(y)) = d_g(x_0,y)$; therefore, the change of variables $y = \sigma^{-1}(x)$ implies that

$$\begin{aligned}
\int_M \frac{(\sigma u)^2(x)}{d_g^2(x_0,x)}\mathrm{d}v_g(x) &= \int_M \frac{u^2(\sigma^{-1}(x))}{d_g^2(x_0,x)}\mathrm{d}v_g(x) = \int_M \frac{u^2(y)}{d_g^2(x_0,\sigma(y))}\mathrm{d}v_g(\sigma(y)) \\
&= \int_M \frac{u^2(y)}{d_g^2(x_0,y)}\mathrm{d}v_g(y).
\end{aligned}$$

In particular, the functional $u \mapsto \mathcal{N}_\mu(u)$ is G-invariant on $H^1(M)$.

(iii) Furthermore, since $\alpha : M \to \mathbb{R}$ depends only on $d_g(x_0, \cdot)$, it is also G-invariant, and one can prove by a change of variables that for every $\sigma \in G$ and $u \in H^1(M)$,

$$\begin{aligned}
\mathcal{F}^+(\sigma u) &= \int_M \alpha(x) F^+((\sigma u)(x)) \mathrm{d}v_g(x) = \int_M \alpha(x) F^+(u(\sigma^{-1}(x))) \mathrm{d}v_g(x) \\
&= \int_M \alpha(\sigma(y)) F^+(u(y)) \mathrm{d}v_g(\sigma(y)) = \int_M \alpha(y) F^+(u(y)) \mathrm{d}v_g(y) \\
&= \mathcal{F}^+(u),
\end{aligned}$$

i.e., $\mathcal{F}^+$ is G-invariant on $H^1(M)$. In conclusion, the energy functional $\mathcal{E}^+ = \mathcal{N}_\mu/2 - \lambda \mathcal{F}^+$ is G-invariant on $H^1(M)$. $\square$

We introduce the subspace of G-invariant functions, namely

$$H^1_G(M) := \mathsf{Fix}_{H^1(M)}(G) = \{u \in H^1(M) | \sigma u = u \text{ for all } \sigma \in G\},$$

and the restricted energy functional to $H^1_G(M)$ as $\mathcal{E}^+_G := \mathcal{E}^+|_{H^1_G(M)}$.

Now, we are in the position to use the principle of symmetric criticality, see Proposition 2.4.1. If $u \in H^1_G(M)$ is a critical point of the restricted energy functional $\mathcal{E}^+_G$ then u is also a critical point of the initial energy functional $\mathcal{E}^+$.

5.3.3 Spectral gap estimate for $\mathcal{F}^+/\mathcal{N}_\mu$ on $H^1_G(M)$ (Step 3)

We are going to prove that for every admissible μ from the statement of the theorem, one has

$$0 < \sup_{u \in H^1_G(M) \setminus \{0\}} \frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)} < +\infty. \tag{5.8}$$

Let $q \in (2, 2^*)$ and fix arbitrarily $\epsilon > 0$ together with the number $\delta_\epsilon > 0$ appearing in (5.5). By the boundedness of the function $t \mapsto \frac{\max |\partial F^+(t)|}{t^{q-1}}$ on $[\delta_\epsilon, \delta_\epsilon^{-1}]$ and due to (5.5), there exists $l_\epsilon > 0$ such that

$$0 \le |\xi| \le \epsilon t + l_\epsilon t^{q-1}, \ \forall t \ge 0, \ \xi \in \partial F^+(t) = \partial F(t). \tag{5.9}$$

Note that we have

$$0 \le |F^+(t)| \le \epsilon t^2 + l_\epsilon |t|^q, \ \forall t \in \mathbb{R}. \tag{5.10}$$

Indeed, for $t \ge 0$ we apply Lebourg's mean value theorem together with (5.9), while for $t \le 0$, we have by definition that $F^+(t) = F(0) = 0$, thus the latter relation trivially holds.

Consequently, taking into account the continuous Sobolev embedding, estimate (5.10) shows that for every $u \in H^1_G(M)$ we have

$$\begin{aligned} 0 \leq |\mathcal{F}^+(u)| &= \left| \int_M \alpha(x) F^+(u(x)) \mathrm{d}v_g \right| \leq \int_M \alpha(x) \left| F^+(u(x)) \right| \mathrm{d}v_g \\ &\leq \|\alpha\|_{L^\infty} \left(\epsilon \|u\|^2_{H^1} + l_\epsilon (K^\pm_q)^q \|u\|^q_{H^1} \right), \end{aligned}$$

where $K^\pm_q > 0$ are the continuous embedding constants. Accordingly, for every $u \in H^1_G(M) \setminus \{0\}$ one has that

$$0 \leq \frac{|\mathcal{F}^+(u)|}{\mathcal{N}_\mu(u)} \leq c_\mu^{-1} \|\alpha\|_{L^\infty} \left(\epsilon + l_\epsilon (K^\pm_q)^q \|u\|^{q-2}_{H^1} \right),$$

where $c_\mu > 0$ is the constant from (5.4). Due to the fact that $q > 2$ and $\epsilon > 0$ is arbitrarily fixed, it turns out that

$$\frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)} \to 0 \text{ as } \|u\|_{H^1} \to 0, u \in H^1_G(M). \tag{5.11}$$

The counterpart of (5.11) at 'infinity' reads as

$$\frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)} \to 0 \text{ as } \|u\|_{H^1} \to +\infty, u \in H^1_G(M). \tag{5.12}$$

Indeed, combining the boundedness of $t \mapsto \frac{\max |\partial F^+(t)|}{t^{1/2}}$ on $[\delta_\epsilon, \delta_\epsilon^{-1}]$ with the estimate (5.5), one can find $L_\epsilon > 0$ such that

$$0 \leq |\xi| \leq \epsilon t + L_\epsilon t^{1/2}, \ \forall t \geq 0, \ \xi \in \partial F^+(t) = \partial F(t).$$

Due to hypothesis $(\mathbf{H}_\alpha)$, one has that $\alpha \in L^4(M)$. Then using Lebourg's mean value theorem, continuous embeddings and Hölder's inequality, we can proceed as before, obtaining

$$0 \leq |\mathcal{F}^+(u)| \leq \int_M \alpha(x) \left| F^+(u(x)) \right| \mathrm{d}v_g \leq \epsilon \|\alpha\|_{L^\infty} \|u\|^2_{H^1} + L_\epsilon \|\alpha\|_{L^4} \|u\|^{\frac{3}{2}}_{H^1}. \tag{5.13}$$

Consequently, for every $u \in H^1_G(M) \setminus \{0\}$ we have

$$0 \leq \frac{|\mathcal{F}^+(u)|}{\mathcal{N}_\mu(u)} \leq c_\mu^{-1} \left(\epsilon \|\alpha\|_{L^\infty} + L_\epsilon \|\alpha\|_{L^4} \|u\|^{-\frac{1}{2}}_{H^1} \right).$$

This estimate together with the arbitrariness of $\epsilon > 0$ immediately imply (5.12).

In particular, (5.11) and (5.12) imply that the second inequality in (5.8) holds. In order to check the first inequality in (5.8), we recall by $(\mathbf{H}_3)$ that there exists $t_0^+ > 0$ such that $F(t_0^+) > 0$. Moreover, by $(\mathbf{H}_\alpha)$, since $\alpha \neq 0$ and it depends only on $d_g(x_0, \cdot)$, there exists an open x_0-centered annulus on M with radius $0 \leq r < R$, i.e. $A_{x_0}(r,R) = \{x \in M : r < d_g(x_0,x) < R\}$, such that $\operatorname{essinf}_{A_{x_0}(r,R)}\alpha = \alpha_0 > 0$. For sufficiently small $\epsilon > 0$ (e.g. $\epsilon < (R-r)/3$), we consider the function $w_\epsilon : M \to \mathbb{R}$ defined by

$$w_\epsilon(x) = \begin{cases} \frac{t_0^+}{\epsilon}(d_g(x_0,x) - r) & \text{if } d_g(x_0,x) \in (r, r+\epsilon), \\ t_0^+ & \text{if } d_g(x_0,x) \in [r+\epsilon, R-\epsilon], \\ \frac{t_0^+}{\epsilon}(R - d_g(x_0,x)) & \text{if } d_g(x_0,x) \in (R-\epsilon, R), \\ 0 & \text{if } x \notin A_{x_0}(r,R). \end{cases}$$

Note that $w_\epsilon \in H_G^1(M)$ and $w_\epsilon \geq 0$. Moreover,

$$\begin{aligned} \mathcal{F}^+(w_\epsilon) &= \int_M \alpha(x)F(w_\epsilon(x))\mathrm{d}v_g = \int_{A_{x_0}(r,R)} \alpha(x)F(w_\epsilon(x))\mathrm{d}v_g \\ &\geq \alpha_0 F(t_0^+)V_g(A_{x_0}(r+\epsilon, R-\epsilon)) \\ &\quad -\|\alpha\|_{L^\infty} \max_{t\in[0,t_0^+]} |F(t)|(V_g(A_{x_0}(r,r+\epsilon)) + V_g(A_{x_0}(R-\epsilon,R))). \end{aligned}$$

By continuity reason, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0,\epsilon_0)$,

$$\mathcal{F}^+(w_\epsilon) \geq \alpha_0 F(t_0^+)V_g(A_{x_0}(r,R))/2 > 0.$$

On the other hand, by (5.4) and the eikonal equation ($|\nabla_g d_g(x_0,\cdot)| = 1$ a.e. on M) we have the estimate

$$\mathcal{N}_\mu(w_\epsilon) \leq \|w_\epsilon\|_{H^1}^2 \leq (t_0^+)^2(1+\epsilon^{-2})V_g(A_{x_0}(r,R)) < +\infty.$$

Consequently, it turns out that

$$0 < \frac{\mathcal{F}^+(w_{\epsilon_0/2})}{\mathcal{N}_\mu(w_{\epsilon_0/2})} \leq \sup_{u\in H_G^1(M)\setminus\{0\}} \frac{\mathcal{F}^+(u)}{\mathcal{N}_\mu(u)},$$

which shows the validity of the first inequality in (5.8).

5.3.4 Analytic properties of $\mathcal{E}_G^+$ (Step 4)

We shall prove three basic properties of $\mathcal{E}_G^+$ on $H_G^1(M)$, namely, coercivity and boundedness from below, as well as the validity of the non-smooth Palais-Smale condition.

Let $\lambda > 0$ be arbitrarily fixed and μ be in the admissible range (cf. the statement of the theorem). First, we observe by (5.4) and (5.13) that for every $u \in H_G^1(M)$ we have

$$\begin{aligned} \mathcal{E}_G^+(u) &= \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}^+(u) \\ &\geq \left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}\right)\|u\|_{H^1}^2 - \lambda L_\epsilon\|\alpha\|_{L^4}\|u\|_{H^1}^{\frac{3}{2}}. \end{aligned}$$

In particular, for sufficiently small $\epsilon > 0$, e.g. $0 < \epsilon < \frac{c_\mu}{2}\lambda^{-1}\|\alpha\|_{L^\infty}^{-1}$, it follows that $\mathcal{E}_G^+$ is bounded from below and coercive, i.e., $\mathcal{E}_G^+(u) \to +\infty$ whenever $\|u\|_{H^1} \to +\infty$.

Let $\{u_k\}_k \subset H_G^1(M)$ be a Palais-Smale sequence for $\mathcal{E}_G^+$, i.e., for some $M > 0$, one has $|\mathcal{E}_G^+(u_k)| \leq M$ and $m(u_k) \to 0$ as $k \to \infty$, where $m(u) = \min\{\|\xi\|_* : \xi \in \partial\mathcal{E}_G^+(u)\}$. We want to prove that, up to a subsequence, $\{u_k\}_k$ strongly converges to some element in $H_G^1(M)$. Being $\mathcal{E}_G^+$ coercive, the sequence $\{u_k\}_k \subset H_G^1(M)$ is bounded in $H_G^1(M)$. Therefore, due to the fact that $H_G^1(M)$ can be compactly embedded into $L^q(M)$, $q \in (2, 2^*)$ (see Preliminaries subsection 2.5.5), up to a subsequence, one has that

$$u_k \to u \text{ weakly in } H_G^1(M); \tag{5.14}$$

$$u_k \to u \text{ strongly in } L^q(M),\ q \in (2, 2^*). \tag{5.15}$$

By the definition of $\mathcal{E}_G^+$ we have that

$$(\mathcal{E}_G^+)^0(u_k; u - u_k) = \frac{1}{2}\langle \mathcal{N}_\mu'(u_k), u - u_k\rangle + \lambda(-\mathcal{F}^+)^0(u_k; u - u_k);$$
$$(\mathcal{E}_G^+)^0(u; u_k - u) = \frac{1}{2}\langle \mathcal{N}_\mu'(u), u_k - u\rangle + \lambda(-\mathcal{F}^+)^0(u; u_k - u).$$

Note that

$$\frac{1}{2}\langle \mathcal{N}_\mu'(u_k), u - u_k\rangle + \frac{1}{2}\langle \mathcal{N}_\mu'(u), u_k - u\rangle = -\mathcal{N}_\mu(u_k - u).$$

By adding the above relations it turns out that

$$\begin{aligned}\mathcal{N}_\mu(u_k - u) \quad &= \quad \lambda\left((\mathcal{F}^+)^0(u_k; -u + u_k) + (\mathcal{F}^+)^0(u; -u_k + u)\right) \\ &\quad - (\mathcal{E}_G^+)^0(u_k; u - u_k) - (\mathcal{E}_G^+)^0(u; u_k - u).\end{aligned} \tag{5.16}$$

In the sequel, we are going to estimate the terms in the right hand side of latter expression. First, by inequality (5.6) and (5.9) together with the fact that $\partial F^+(t) = \{0\}$ for $t \leq 0$, we have

$$\begin{aligned}I_k^1 &:= (\mathcal{F}^+)^0(u_k; -u + u_k) + (\mathcal{F}^+)^0(u; -u_k + u) \\ &\leq \int_M \alpha(x)\left[(F^+)^0(u_k(x); u_k(x) - u(x)) + (F^+)^0(u(x); u(x) - u_k(x))\right] \mathrm{d}v_g \\ &= \int_M \alpha(x)\left[\max\{\xi_k(u_k(x) - u(x)) : \xi_k \in \partial F^+(u_k(x))\}\right. \\ &\qquad\qquad \left. + \max\{\xi(u(x) - u_k(x)) : \xi \in \partial F^+(u(x))\}\right] \mathrm{d}v_g \\ &\leq \|\alpha\|_{L^\infty} \int_M [\epsilon(|u_k(x)| + |u(x)|) + l_\epsilon(|u_k(x)|^{q-1} \\ &\qquad\qquad + |u(x)|^{q-1})]|u(x) - u_k(x)| \mathrm{d}v_g \\ &\leq 2\epsilon\|\alpha\|_{L^\infty}(\|u_k\|_{H^1}^2 + \|u\|_{H^1}^2) + l_\epsilon\|\alpha\|_{L^\infty}(\|u_k\|_{L^q}^{q-1} + \|u\|_{L^q}^{q-1})\|u_k - u\|_{L^q}.\end{aligned}$$

By the arbitrariness of $\epsilon > 0$ and the convergence property (5.15), the latter estimate shows that

$$\limsup_{k\to\infty} I_k^1 \leq 0.$$

Let $\xi_k \in \partial\mathcal{E}_G^+(u_k)$ be such that $m(u_k) = \|\xi_k\|_*$. Thus, we have that

$$I_k^2 := (\mathcal{E}_G^+)^0(u_k; u - u_k) \geq \langle \xi_k, u - u_k \rangle \geq -\|\xi_k\|_* \|u - u_k\|_{H^1}.$$

Consequently, since $m(u_k) = \|\xi_k\|_* \to 0$ as $k \to \infty$, we have that

$$\liminf_{k\to\infty} I_k^2 \geq 0.$$

Moreover, for every $\xi \in \partial\mathcal{E}_G^+(u)$, we also have that $I_k^3 := (\mathcal{E}_G^+)^0(u; u_k - u) \geq \langle \xi, u_k - u \rangle$; thus, by the weak convergence property (5.14) we have that

$$\liminf_{k\to\infty} I_k^3 \geq 0.$$

Now, combining estimates above with relation (5.16), we have that

$$0 \leq \limsup_{k\to\infty} \mathcal{N}_\mu(u_k - u) \leq \limsup_{k\to\infty} I_k^1 - \liminf_{k\to\infty} I_k^2 - \liminf_{k\to\infty} I_k^3 \leq 0,$$

i.e., $\mathcal{N}_\mu(u_k - u) \to 0$ as $k \to \infty$. Due to (5.4), it turns out that $u_k \to u$ strongly in the H^1-norm as $k \to \infty$, which is the desired property. $\square$

5.3.5 Local minimum point for $\mathcal{E}_G^+$: first solution (Step 5)

Let

$$\lambda_0^+ := \inf_{\substack{u \in H_G^1(M) \\ \mathcal{F}^+(u) > 0}} \frac{\mathcal{N}_\mu(u)}{2\mathcal{F}^+(u)}.$$

Due to Step 3, see (5.8), one has that $0 < \lambda_0^+ < \infty$.

If we fix $\lambda > \lambda_0^+$, one can find $\tilde{w}_\lambda \in H_G^1(M)$ with $\mathcal{F}^+(\tilde{w}_\lambda) > 0$ such that

$$\lambda > \frac{\mathcal{N}_\mu(\tilde{w}_\lambda)}{2\mathcal{F}^+(\tilde{w}_\lambda)} \geq \lambda_0^+.$$

Thus, by the latter inequality we have

$$C_\lambda^1 := \inf_{H_G^1(M)} \mathcal{E}_G^+ \leq \mathcal{E}_G^+(\tilde{w}_\lambda) = \frac{1}{2}\mathcal{N}_\mu(\tilde{w}_\lambda) - \lambda\mathcal{F}^+(\tilde{w}_\lambda) < 0.$$

Due to the fact that $\mathcal{E}_G^+$ is bounded from below and verifies the non-smooth Palais-Smale condition (see Step 4), C_λ^1 is a critical value of $\mathcal{E}_G^+$, see Theorem 2.4.1, in particular there exists $u_\lambda^1 \in H_G^1(M)$ such that $\mathcal{E}_G^+(u_\lambda^1) = C_\lambda^1 < 0$ and $0 \in \partial\mathcal{E}_G^+(u_\lambda^1)$. Since $\mathcal{E}_G^+(u_\lambda^1) < 0 = \mathcal{E}_G^+(0)$, it turns out that $u_\lambda^1 \neq 0$. Although we just have continuous Sobolev embeddings on $H^1(M)$, the lack of compactness is compensated by isometric actions, and applying principle of symmetric criticality imply that u_λ^1 is a critical point also for the initial energy functional (see Step 2), i.e., $0 \in \partial\mathcal{E}^+(u_\lambda^1)$. According to (the final part of) Step 1, $u_\lambda^1 \in H_G^1(M)$ is a positive G-invariant weak solution to the differential inclusion $(\mathcal{D})$.

5.3.6 Minimax-type critical point for $\mathcal{E}_G^+$: second solution (Step 6)

Let $\lambda > \lambda_0^+$. Due to (5.10), for sufficiently small $\epsilon > 0$ (e.g., $\frac{c_\mu}{4} > \epsilon\lambda\|\alpha\|_{L^\infty}$) and for every $u \in H_G^1(M)$ one has that

$$\begin{aligned}\mathcal{E}_G^+(u) &= \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}^+(u)\\ &\geq \left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}\right)\|u\|_{H^1}^2 - \lambda\|\alpha\|_{L^\infty} l_\epsilon (K_q^\pm)^q \|u\|_{H^1}^q,\end{aligned}$$

where $q \in (2, 2^*)$ and $K_q^\pm > 0$ are the embedding constants from (2.5) and (2.8), respectively. Let

$$\rho_\lambda = \min\left\{\|\tilde{w}_\lambda\|_{H^1}, \left(\frac{\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}}{2\lambda\|\alpha\|_{L^\infty} l_\epsilon (K_q^\pm)^q}\right)^{\frac{1}{q-2}}\right\}.$$

The choice of $\rho_\lambda > 0$ and Step 4 show that

$$\inf_{\|u\|_{H^1} = \rho_\lambda; u \in H_G^1(M)} \mathcal{E}_G^+(u) \geq \frac{1}{2}\left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}\right)\rho_\lambda^2 > 0 = \mathcal{E}_G^+(0) > \mathcal{E}_G^+(\tilde{w}_\lambda).$$

The latter estimate shows that the functional $\mathcal{E}_G^+$ has the mountain pass geometry. On account of Step 4, since $\mathcal{E}_G^+$ satisfies the non-smooth Palais-Smale condition, we may apply the mountain pass theorem with $\gamma(0) = 0$ and $\gamma(1) = \tilde{w}_\lambda$, see 2.4.2, guaranteeing the existence of critical point $u_\lambda^2 \in H_G^1(M)$, such that $0 \in \partial\mathcal{E}_G^+(u_\lambda^2)$. Since $\mathcal{E}_G^+(u_\lambda^2) > 0$, $u_\lambda^2 \neq u_\lambda^1$. The rest of the proof is similar to the end of Step 5, which shows that $u_\lambda^2 \in H_G^1(M)$ is indeed a positive, G-invariant weak solution to the differential inclusion $(\mathcal{D})$, different from u_λ^1.

Naturally we may study the case with similar steps as before, whenever the problem $(\mathcal{D})$ is considered with $F^-(t) = F(t^-)$, where t^- is the non-positive part of t.

5.3.7 Repetition of Steps 1-6 for $\mathcal{E}^-$ (Step 7)

Let $F^-(t) = F(t_-), t \in \mathbb{R}$, where $t_- = \min(t, 0)$. The locally Lipschitz energy functional $\mathcal{E}^- : H^1(M) \to \mathbb{R}$ is defined as

$$\mathcal{E}^-(u) = \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}^-(u),$$

where

$$\mathcal{F}^-(u) = \int_M \alpha(x)F^-(u(x))\mathrm{d}v_g.$$

One can show that if $u \in H^1(M)$ is a critical point of $\mathcal{E}^-$, i.e., $0 \in \partial\mathcal{E}^-(u)$, then it is a non-positive solution of $(\mathcal{D})$, cf. Step 1.

One can prove in a similar way as in section 5.3 that $\mathcal{F}^-$ is G-invariant on $H^1_G(M)$, and if $u \in \mathsf{Fix}_{H^1(M)}(G) =: H^1_G(M)$ is a critical point of $\mathcal{E}^-_G := \mathcal{E}^-|_{H^1_G(M)}$ then $0 \in \partial\mathcal{E}^-(u)$ as well, cf. Step 2.

Instead of the spectral gap estimate (5.8), one can prove

$$0 < \sup_{u \in H^1_G(M)\setminus\{0\}} \frac{\mathcal{F}^-(u)}{\mathcal{N}_\mu(u)} < +\infty,$$

cf. Step 3, and similar analytic properties are valid for $\mathcal{E}^-_G$ as in Step 4 (i.e, coercivity, boundedness from below, and the validity of the non-smooth Palais-Smale condition). Here, we use again the compact embedding.

Finally, if

$$\lambda_0^- := \inf_{\substack{u \in H^1_G(M) \\ \mathcal{F}^-(u) > 0}} \frac{\mathcal{N}_\mu(u)}{2\mathcal{F}^-(u)},$$

by the previous part we know that $0 < \lambda_0^- < \infty$ and similarly to Steps 5 and 6, we can guarantee for every $\lambda > \lambda_0^-$ a local minimum point $u^3_\lambda \in H^1_G(M)$ of $\mathcal{E}^-_G$ with $\mathcal{E}^-_G(u^3_\lambda) < 0$ and a minimax-type point $u^4_\lambda \in H^1_G(M)$ of $\mathcal{E}^-_G$ with $\mathcal{E}^-_G(u^4_\lambda) > 0$; in particular, $u^3_\lambda \neq u^4_\lambda$ and none of them is trivial. These elements are also G-invariant, non-positive solutions to the differential inclusion $(\mathcal{D})$.

If we choose $\lambda_0 = \max(\lambda_0^+, \lambda_0^-)$, we can apply the above arguments, providing four different, non-zero G-invariant solutions to the differential inclusion $(\mathcal{D})$ for every $\lambda > \lambda_0$, two of them being non-negative and the other two being non-positive. The proof is complete. $\square$

5.4 Super-quadratic case: proof of Theorem 5.1.3

We assume in the sequel that the hypotheses of Theorem 5.1.3 are fulfilled. We again divide the proof into some steps.

5.4.1 Functional setting (Step 1)

In view of the previous section, this part is standard. Indeed, the energy functional $\mathcal{E}$: $H^1(M) \to \mathbb{R}$ is defined as

$$\mathcal{E}(u) = \frac{1}{2}\mathcal{N}_\mu(u) - \lambda\mathcal{F}(u),$$

where

$$\mathcal{F}(u) = \int_M \alpha(x)F(u(x))\mathrm{d}v_g.$$

Note that by $(\mathbf{H}_1)$ and $(\mathbf{H}_5)$, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$|\xi| \leq \epsilon|t| + C_\epsilon|t|^{q-1}, \ \forall t \in \mathbb{R}, \ \xi \in \partial F(t). \tag{5.17}$$

Consequently, one has

$$|F(t)| \leq \epsilon t^2 + C_\epsilon|t|^q, \ \forall t \in \mathbb{R}. \tag{5.18}$$

Since $2 < q < 2 + \frac{4}{n} < 2^*$, by using Lebourg's mean value theorem and (5.17), one can prove that $\mathcal{F}$ is well-defined and locally Lipschitz on $H^1(M)$. It is now standard to show that any critical point $u \in H^1(M)$ of $\mathcal{E}$ is a solution of $(\mathcal{D})$.

5.4.2 Isometry actions (Step 2)

One can prove in a similar way as in section 5.3 that $\mathcal{E}$ is G-invariant on $H^1(M)$. Moreover, the principle of symmetric criticality (Proposition 2.4.1) implies that if $u \in \mathsf{Fix}_{H^1(M)}(G) =: H^1_G(M)$ is a critical point of $\mathcal{E}_G := \mathcal{E}|_{H^1_G(M)}$ then u is also a critical point of $\mathcal{E}$.

5.4.3 Super-quadracity of F at infinity (Step 3)

We are going to prove that

$$F(t) \geq \frac{C}{\nu - 2}|t|^\nu, \ \forall t \in \mathbb{R}, \tag{5.19}$$

where $\nu > 2$ and $C > 0$ come from hypothesis $(\mathbf{H}_4)$; this means in particular that F is super-quadratic at infinity (as $\nu > 2$). To do this, let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(t) = t^{-2}F(t) - \frac{C}{\nu - 2}|t|^{\nu-2},\ t \neq 0,$$

and $h(0) = 0$. Note that h is well-defined and locally Lipschitz (indeed, by $(\mathbf{H}_1)$ and $F(0) = 0$ we have that $F(t) = o(t^2)$ as $t \to 0$). One has

$$\partial h(t) = -2t^{-3}F(t) + t^{-2}\partial F(t) - C|t|^{\nu-4}t,\ \forall t \in \mathbb{R} \setminus \{0\}.$$

We shall prove (5.19) for $t \geq 0$, the case $t \leq 0$ being similar. Whenever $t = 0$, (5.19) clearly holds. Let $t > 0$; then by Lebourg's mean value theorem, there exist $\theta \in (0, t)$ and $\xi_h \in \partial h(\theta)$ such that $h(t) = h(t) - h(0) = \xi_h t$. In turn, there exists $\xi_F \in \partial F(\theta)$ such that $\xi_h = -2\theta^{-3}F(\theta) + \theta^{-2}\xi_F - C\theta^{\nu-3}$ and by $(\mathbf{H}_4)$ we have that

$$\begin{aligned}
h(t) = \xi_h t &= (-2\theta^{-3}F(\theta) + \theta^{-2}\xi_F - C\theta^{\nu-3})t \\
&= -\theta^{-3}(2F(\theta) + \xi_F(-\theta) + C\theta^{\nu})t \\
&\geq -\theta^{-3}(2F(\theta) + F^0(\theta; -\theta) + C\theta^{\nu})t \\
&\geq 0,
\end{aligned}$$

which concludes the proof. In particular, combining (5.18) with (5.19), we necessarily have that

$$\nu \leq q. \tag{5.20}$$

5.4.4 Non-smooth Cerami compactness condition for $\mathcal{E}_G$ (Step 4)

Let $\{u_k\}_k \subset H^1_G(M)$ be a Cerami sequence for $\mathcal{E}_G$, i.e., for some $M > 0$, one has $|\mathcal{E}_G(u_k)| \leq M$ and $(1 + \|u_k\|_{H^1})m(u_k) \to 0$ as $k \to \infty$, where $m(u) = \min\{\|\xi\|_* : \xi \in \partial\mathcal{E}_G(u)\}$. Our objective is to prove that, up to a subsequence, $\{u_k\}_k$ strongly converges to some element in $H^1_G(M)$.

To do that we first prove the following lemma.

Lemma 5.4.1. *Under the assumptions of the theorem 5.1.3 the Cerami sequence $\{u_k\}_k \subset H^1_G(M)$ of $\mathcal{E}_G(u)$ is bounded in $L^\nu(M)$.*

Proof. For every $k \in \mathbb{N}$, let $\xi_k \in \partial\mathcal{E}_G(u_k)$ be such that $\|\xi_k\|_* = m(u_k)$. We observe that

$$\mathcal{E}^0_G(u_k; u_k) \geq \langle \xi_k, u_k \rangle \geq -\|\xi_k\|_*\|u_k\|_{H^1} \geq -(1 + \|u_k\|_{H^1})m(u_k).$$

Since $(1+\|u_k\|_{H^1})m(u_k) \to 0$ as $k \to \infty$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ one has that $\mathcal{E}_G^0(u_k; u_k) \geq -1$. Consequently, inequality (5.6) (which is also valid due to (5.17)) and hypothesis $(\mathbf{H}_4)$ imply for every $k \in \mathbb{N}$ that

$$\begin{aligned} 2M+1 &\geq 2\mathcal{E}_G(u_k) - \mathcal{E}_G^0(u_k; u_k) \\ &= \mathcal{N}_\mu(u_k) - 2\lambda\mathcal{F}(u_k) - \frac{1}{2}\langle \mathcal{N}_\mu'(u_k); u_k\rangle - \lambda(-\mathcal{F})^0(u_k; u_k) \\ &= -\lambda\left(2\mathcal{F}(u_k) + \mathcal{F}^0(u_k; -u_k)\right) \\ &\geq -\lambda\int_M \alpha(x)\left(2F(u_k(x)) + F^0(u_k(x); -u_k(x))\right)\mathrm{d}v_g \\ &\geq \lambda C\int_M \alpha(x)|u_k(x)|^\nu \mathrm{d}v_g. \end{aligned}$$

Since $\alpha \in L^\infty(M)$ and $\operatorname{essinf}_{x\in M}\alpha(x) = \alpha_0 > 0$, the latter estimate implies that

$$2M+1 \geq \lambda C\alpha_0\|u_k\|_{L^\nu}^\nu, \ \forall k \in \mathbb{N}.$$

Consequently, $\{u_k\}_k$ is bounded in $L^\nu(M)$. □

Now we are in the postion to prove the following lemma.

Lemma 5.4.2. *Under the assumptions of the theorem 5.1.3 the Cerami sequence $\{u_k\}_k \subset H_G^1(M)$ of $\mathcal{E}_G(u)$ is bounded in $H_G^1(M)$.*

Proof. By (5.18), for every small $\epsilon > 0$ there exists $\tilde{C}_\epsilon > 0$ such that for every $k \in \mathbb{N}$,

$$\begin{aligned} M \geq \mathcal{E}_G(u_k) &= \frac{1}{2}\mathcal{N}_\mu(u_k) - \lambda\mathcal{F}(u_k) \\ &\geq \left(\frac{c_\mu}{2} - \epsilon\lambda\|\alpha\|_{L^\infty}\right)\|u_k\|_{H^1}^2 - \lambda\tilde{C}_\epsilon\|\alpha\|_{L^\infty}\|u_k\|_{L^q}^q. \end{aligned}$$

In particular, if $\frac{c_\mu}{4} > \epsilon\lambda\|\alpha\|_{L^\infty}$, then there exists $M_\epsilon > 0$ and $\overline{C}_\epsilon > 0$ such that

$$\|u_k\|_{H^1}^2 \leq M_\epsilon + \overline{C}_\epsilon\|u_k\|_{L^q}^q, \ \forall k \in \mathbb{N}. \tag{5.21}$$

On account of (5.20), we distinguish two cases:

a) $\nu = q$. Since $\{u_k\}_k$ is bounded in $L^\nu(M)$ and $\nu = q$, by (5.21) we also have that $\{u_k\}_k$ is bounded in $H_G^1(M)$.

b) $\nu < q$. Let $\eta \in (0,1)$ be such that $\frac{1}{q} = \frac{1-\eta}{\nu} + \frac{\eta}{2^*}$. By (5.21) and a standard interpolation inequality we have that

$$\begin{aligned} \|u_k\|_{H^1}^2 &\leq M_\epsilon + \overline{C}_\epsilon\|u_k\|_{L^q}^q \\ &\leq M_\epsilon + \overline{C}_\epsilon\|u_k\|_{L^\nu}^{(1-\eta)q}\|u_k\|_{L^{2^*}}^{\eta q} \\ &\leq M_\epsilon + \overline{C}_\epsilon(K_q^\pm)^{\eta q}\|u_k\|_{L^\nu}^{(1-\eta)q}\|u_k\|_{H^1}^{\eta q}, \end{aligned} \tag{5.22}$$

where $K_q^{\pm} > 0$ are the embedding constant. Since $q < 2+\frac{4}{n}$, we have that $\nu > 2 > \frac{n(q-2)}{2}$. We observe that $\nu > \frac{n(q-2)}{2}$ together with $\frac{1}{q} = \frac{1-\eta}{\nu} + \frac{\eta}{2^*}$ is equivalent to $\eta q < 2$. The latter inequality and (5.22) imply that $\{u_k\}_k$ is bounded in $H_G^1(M)$. $\square$

Now, we can proceed as in Step 4, see section 5.3; in this way we conclude that $\{u_k\}_k$ strongly converges (up to a subsequence) to some element in $H_G^1(M)$.

5.4.5 Existence/multiplicity of critical points for $\mathcal{E}_G$ (Step 5)

Under the assumptions of the theorem, one can prove as above that $\mathcal{E}_G$ has the mountain pass geometry. By Step 4 and on account of the mountain pass theorem for locally Lipschitz functions, see e.g. Kourogenis and Papageorgiou [21], we conclude the existence of a non-zero critical point for $\mathcal{E}_G$. When F is even, we may apply the non-smooth fountain theorem involving the Cerami compactness condition, see e.g. Kristály [23], guaranteeing the existence of a sequence of critical points for the functional $\mathcal{E}_G$. All these points are G-invariant solutions to the differential inclusion $(\mathcal{D})$, which concludes the proof. $\square$

Chapter 6

Schrödinger-Maxwell differential inclusion system

Electrostatic variations of Schrödinger-Maxwell systems have been subject of several investigations, see Kristály, Repovš [32], Azzollini and Pimenta [5], describing a charged quantum-mechanical particle interacting with the electromagnetic field:

$$\begin{cases} -\Delta u(x) + eu(x)\phi(x) = f(u(x)), & x \in \mathbb{R}^3; \\ -\Delta\phi(x) = eu^2(x), & x \in \mathbb{R}^3, \end{cases}$$

where Δ denotes the Laplace operator, $\phi : \mathbb{R}^3 \to \mathbb{R}$ is the electric potential, e is the electron charge constant, while the function $u : \mathbb{R}^3 \to \mathbb{R}$ is the field associated to the particle. Recent researches focus to curved spaces, see e.g. Farkas and Kristály ([17]), where existence results for Schrödinger-Maxwell systems are provided on non-compact Hadamard manifolds, involving sublinear or oscillatory terms at infinity.

Considering a broad class of non-compact Riemannian manifolds, we study a non-smooth Schrödinger-Maxwell inclusion system, equipped with a nonlinear term on the non-compact Riemannian manifold (M, g), namely

$$\begin{cases} -\Delta_g u(x) + u(x) + u(x)\phi(x) \in \lambda\alpha(x)\partial F(u(x)), & x \in M; \\ -\Delta_g\phi(x) + \phi(x) = 4\pi u^2(x), & x \in M, \end{cases} \qquad (\mathcal{S}M)$$

where Δ_g denotes the Laplace-Beltrami operator on (M, g), $\lambda > 0$ is a parameter, and the unknown terms $u, \phi : M \to \mathbb{R}$. In the sequel, $\alpha : M \to \mathbb{R}$ is a potential, and ∂F stands for the generalized gradient of the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ in the sense of Clarke (see [12]), satisfying the following conditions:

$(\mathbf{F}_1)$: $\lim_{t\to 0+} \frac{\max\{|\theta|:\theta\in\partial F(t)\}}{t} = 0$;

$(\mathbf{F}_2)$: $\lim_{t\to \infty} \frac{\max\{|\theta|:\theta\in\partial F(t)\}}{t} = 0$;

$(\mathbf{F}_3)$: $F(0) = 0$;

$(\mathbf{C}_\alpha)$: The function $\alpha : M \to \mathbb{R}$ is radially symmetric with respect to some $x_0 \in M$ and $\inf_M \alpha > 0$.

The main objective is to prove non-existence and existence results depending on the parameter λ for the inclusion system $(\mathcal{S}M)$, equipped with non-linear term, not only on Hadamard manifolds, but also on Riemann manifolds with non-negative Ricci curvature. It turns out, similarly to the chapter 5, that the lack of compactness has to be compensated by applying isometric actions and the principle of symmetric criticality in order to use variational methods. Similarly to Farkas and Kristály [17], due to the coupled system we have to introduce a "single variable" energy functional. Our aim is to extend the result of Farkas and Kristály [17] to non-smooth functions and examine the problem on two different classes of curved spaces.

Section 6.1 is devoted to present our main results, while in section 6.2 we prove Theorem 6.1.1. This chapter is based on results proved in Szilák [49].

6.1 Main results

A pair $(u, \phi) \in H^1(M) \times H^1(M)$ is a *weak solution* of the inclusion system $(\mathcal{S}M)$ if there exists a measurable mapping $x \mapsto \xi \in \partial F(u(x))$ such that for all test-functions $v, \psi \in H^1(M)$ one has

$$\int_M \nabla_g u(x)\nabla_g v(x)\mathrm{d}v_g + \int_M u(x)v(x)\mathrm{d}v_g + \int_M u(x)\phi(x)v(x)\mathrm{d}v_g$$
$$= \lambda \int_M \alpha(x)\xi v(x)\mathrm{d}v_g$$

and

$$\int_M \nabla_g u(x)\nabla_g \psi(x)\mathrm{d}v_g + \int_M \phi(x)\psi(x)\mathrm{d}v_g = 4\pi \int_M u^2(x)\psi(x)\mathrm{d}v_g.$$

6.1.1 Maxwell equation

Let us consider the Maxwell equation, namely

$$-\Delta_g \phi(x) + \phi(x) = 4\pi u^2(x), \quad x \in M.$$

A straightforward extension of Kristály and Repovš result (see [32]), based on the Lax-Milgram theorem (see [10]), implies that the equation above admits the existence of the unique solution $\phi_u : H^1(M) \to H^1(M)$ for all $u \in H^1(M)$.

Note, that $\|\phi_u\|^2 = \int_M \phi_u u^2 \mathrm{d}v_g$, $\phi_u \geq 0$; and $u \mapsto \int_M \phi_u u^2 \mathrm{d}v_g$ is convex.

Our main theorem can be stated as follows:

Theorem 6.1.1. (Szilák [49]) *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and G be a compact connected subgroup of $\mathsf{Isom}_g(M)$, $x_0 \in M$ is fixed, and $\mathrm{Fix}_M(G) = \{x_0\}$. Let assume that the potential α and the locally Lipschitz function F satisfy hypotheses ($\mathbf{C}_\alpha$), ($\mathbf{F}_1$)-($\mathbf{F}_3$), respectively. Moreover, let assume that one of the following curvature assumptions holds:*

(a) (M, g) *is Cartan-Hadamard type,*

(b) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mathsf{AVR}_{(M,g)} > 0$ *and G is coercive.*

Then there exist $\lambda_1 > \lambda_0 > 0$ such that the differential inclusion system $(\mathcal{S}M)$ has only the trivial solution for $0 < \lambda < \lambda_0$, and has two different non-trivial solutions whenever $\lambda > \lambda_1$.

6.2 Proof of Theorem 6.1.1

6.2.1 Energy functionals

The energy functional $\mathcal{L} : H^1(M) \times H^1(M) \to \mathbb{R}$ associated to the inclusion system $(\mathcal{S}M)$ is defined as

$$\begin{aligned}\mathcal{L}(u, \phi) = &\frac{1}{2}\|u\|^2_{H^1} + \frac{1}{2}\int_M \phi(x)u^2(x)\mathrm{d}v_g \\ &- \lambda \int_M \alpha(x)F(u(x))\mathrm{d}v_g - \frac{1}{16\pi}\int_M |\nabla_g \phi(x)|^2 \mathrm{d}v_g \\ &- \frac{1}{16\pi}\int_M |\phi(x)|^2 \mathrm{d}v_g.\end{aligned}$$

We note that the energy functional is well defined and locally Lipschitz. Since our objective is to apply variational methods and we are dealing with an inclusion system $(\mathcal{S}M)$, similarly to Farkas and Kristály [17], we define a "single-variable" locally Lipschitz energy functional $\mathcal{E}_\lambda : H^1(M) \to \mathbb{R}$ to that, namely

$$\mathcal{E}_\lambda(u) = \mathcal{N}(u) - \lambda\mathcal{F}(u),$$

where

$$\mathcal{N}(u) = \frac{1}{2}\|u(x)\|_{H^1}^2 + \frac{1}{2}\int_M \phi_u(x)u^2(x)\mathrm{d}v_g,$$

$$\mathcal{F}(u) = \int_M \alpha(x)F(u(x))\mathrm{d}v_g.$$

6.2.2 $\mathcal{E}_\lambda$ is G-invariant

Recalling again subsection 2.4.2, in order to be able to use the principle of symmetric criticality, we claim that the energy functional $\mathcal{E}_\lambda$ is G-invariant on $H^1(M)$. Applying an appropriate change of variable $y = \sigma^{-1}(x)$, for all $\sigma \in G$ and $u \in H^1(M)$ we have that $\|\sigma u\|_{H^1} = \|u\|_{H^1}$, see (5.7). By $(\mathbf{C}_\alpha)$, the function α is radially symmetric. Thus, applying again the change of variable $y = \sigma^{-1}(x)$, for all $\sigma \in G$ and $u \in H^1(M)$ we have

$$\begin{aligned}\mathcal{F}((\sigma u)(x)) &= \int_M \alpha(x)\partial F((\sigma u)(x))\mathrm{d}v_g(x) = \int_M \alpha(x)\partial F(\sigma^{-1}(x))\mathrm{d}v_g(x)\\ &= \int_M \alpha(y)\partial F(y)\mathrm{d}v_g(y)\end{aligned}$$

and it follows that $\mathcal{F}$ is G-invariant.

Applying again the change of variables $y = \sigma^{-1}(x)$, one can prove for all $\sigma \in G$ and $u \in H^1(M)$ that $\phi_{\sigma u}(\sigma(x)) = \phi_u(x)$ and we have

$$\int_M \phi_{\sigma u}(x)(\sigma u)^2\mathrm{d}v_g(x) = \int_M \phi_u(y)u^2(y)\mathrm{d}v_g(y),$$

which proves that $\int_M \phi_u u^2 \mathrm{d}v_g$ is also G-invariant.

Combining these facts, our claim follows. $\square$

Similarly to chapter 5, we introduce again the subspace of G-invariant functions, namely

$$H_G^1(M) := \mathsf{Fix}_{H^1(M)}(G) = \{u \in H^1(M) | \sigma u = u \text{ for all } \sigma \in G\}$$

and also the restricted single variable energy functional onto $H_G^1(M)$ as $\mathcal{E}_{\lambda,G} := \mathcal{E}_\lambda|_{H_G^1(M)}$.

6.2.3 Analytic properties of $\mathcal{E}_{\lambda,G}(u)$

In this subsection we show some basic properties of the single variable energy functional $\mathcal{E}_{\lambda,G}(u)$, namely boundnesses from below, coercivity and the non-smooth Palais-Smale condition.

Based on $(\mathbf{F}_2)$, we can find $\delta > 0$ for every $\varepsilon > 0$ such that if $s \geq \delta > 0$, then $|\xi| \leq \varepsilon s$ where $\xi \in \partial F(s)$. By $(\mathbf{F}_3)$ we have $F(0) = 0$. Thus, an appropriate continuous Sobolev embedding and the Lebourg's mean value theorem imply that

$$
\begin{aligned}
\mathcal{F}(u) &= \int_{u\leq\delta} \alpha(x)F(u(x))\mathrm{d}v_g + \int_{u>\delta} \alpha(x)F(u(x))\mathrm{d}v_g \\
&\leq \|\alpha\|_{L^1(M)} \max_{s<\delta} |F(s)| + \varepsilon\|\alpha\|_{L^\infty(M)}\|u\|^2_{H^1}.
\end{aligned}
$$

Accordingly, the estimate

$$
\begin{aligned}
\mathcal{E}_{\lambda,G}(u) &= \mathcal{N}(u) - \lambda\mathcal{F}(u) \\
&= \frac{1}{2}\|u\|^2_{H^1} + \frac{1}{2}\int_M \phi_u(x)u^2(x)\mathrm{d}v_g - \lambda\mathcal{F}(u(x)) \\
&\geq \|u\|^2_{H^1}\big(\frac{1}{2} - \lambda\varepsilon\|\alpha\|_{L^\infty(M)}\big) - \lambda\|\alpha\|_{L^1(M)} \max_{s\leq\delta}|F(s)|
\end{aligned}
$$

shows that if ε is small enough, more precisely $\varepsilon < \frac{1}{2\lambda\|\alpha\|_{L^\infty(M)}}$, then the energy functional $\mathcal{E}_{\lambda,G}$ is bounded from below and coercive, i.e., $\mathcal{E}_{\lambda,G}(u) \to +\infty$ whenever $\|u\|_{H^1} \to +\infty$.

6.2.4 $\mathcal{E}_{\lambda,G}$ satisfies the non-smooth Palais-Smale condition on $H^1_G(M)$

Based on $(\mathbf{F}_1)$ and $(\mathbf{F}_2)$, for every $\varepsilon > 0$ one has two numbers $\delta_1, \delta_2 > 0$, such that

$$|\xi| \leq \varepsilon s \text{ for all } \xi \in \partial F(s), \forall 0 < s < \delta_1 \text{ and } s > \delta_2.$$

Let us fix $\varepsilon > 0$ together with the numbers δ_1 and δ_2. By assumptions $(\mathbf{F})_1$ and $(\mathbf{F})_2$, it turns out that there exists $l_\varepsilon > 0$ such that for some $q \in (2, 2^*)$ we have

$$0 \leq |\xi| \leq \varepsilon s + l_\varepsilon s^{q-1}, \forall s \geq 0, \xi \in \partial F(s). \tag{6.1}$$

Let $\{u_k\}_k \subset H^1_G(M)$ be a Palais-Smale sequence, i.e. $\mathcal{E}_{\lambda,G}(u_k)$ is bounded and $m(u_k) \to 0$ as $k \to 0$, where $m(u_k) = \min\{\|\xi\|_* : \xi \in \partial\mathcal{E}_{\mu,\lambda}(u_k)\}$. Due to the coerciveness of $\mathcal{E}_{\lambda,G}$, it turns out that $\{u_k\} \subset H^1_G(M)$ is bounded. Thus, the Sobolev compact embeddings (see Preliminaries subsection 2.5.5) imply that up to a subsequent, $\{u_k\}_k$ converges weakly in $H^1(M)$ and strongly in $L^q(M)$, $q \in (2, 2^*)$.

Let us consider the generalized directional derivative of the energy functional $\mathcal{E}_{\lambda,G}$ for all $u, v \in H^1(M)$:

$$\mathcal{E}^\circ_{\lambda,G}(u,v) = \frac{1}{2}\langle\mathcal{N}'(u); v\rangle + \lambda(-\mathcal{F})^\circ(u,v).$$

Applying the latter inequality with parameters $u = u_k$ and $v = u - u_k$, then $u = u$ and $v = u_k - u$ one has

$$\begin{aligned}\frac{1}{2}\|u_k - u\|_{H^1}^2 = &-\frac{1}{2}\int_M \phi_{u_k-u}(u_k-u)^2(x)\mathrm{d}v_g \\ &+ \lambda\mathcal{F}^\circ(u_k; u_k-u) + \lambda\mathcal{F}^\circ(u; u-u_k) \\ &- \mathcal{E}^\circ_{\lambda,G}(u_k; u-u_k) - \mathcal{E}^\circ_{\lambda,G}(u; u_k-u).\end{aligned}$$

On one hand, since $m(u_k) \to 0$, it follows that

$$\lim_{k\to\infty}\inf\left(\mathcal{E}^\circ_{\lambda,G}(u, u_k-u) + \mathcal{E}^\circ_{\lambda,G}(u_k, u-u_k)\right) \geq 0.$$

On the other hand, for all measurable mappings $x \mapsto \xi \in \partial F(u(x))$ and $x \mapsto \xi_k \in \partial F(u_k(x))$ we have the following estimate

$$\begin{aligned}S_k &:= \lambda\mathcal{F}^\circ(u_k; u_k-u) + \lambda\mathcal{F}^\circ(u; u-u_k) \\ &\leq \lambda\int \alpha(x)\left[F^\circ(u_k; u_k-u) + F^\circ(u; u-u_k)\right]\mathrm{d}v_g \\ &= \lambda\int \alpha(x)\left[\max\{\xi_k(u_k-u)\} + \max\{\xi(u-u_k)\}\right]\mathrm{d}v_g \\ &\leq \varepsilon(\|u_k\|_{H^1} + \|u\|_{H^1})\|u_k-u\|_{H^1} + l_\varepsilon(\|u_k\|_{L^q}^{q-1} + \|u\|_{L^q}^{q-1})\|u_k-u\|_{L^q}\end{aligned}$$

where l_ε is defined in the equation (6.1).

Thus, by the arbitrariness of $\varepsilon > 0$ and $u_k \to u$ in $L^q(M)$, as $k \to \infty$ where $q \in (2, 2^*)$, one has that $\limsup_{k\to\infty} S_k \leq 0$. Since $\phi_{u_k-u}(u_k-u)^2 \geq 0$, it follows that $u_k \to u$ in $H^1_G(M)$, which ends the proof. □

6.2.5 Relations between energy functionals

On one hand, assuming that the pair $(u, \phi) \in H^1(M) \times H^1(M)$ is a critical point of the locally Lipschitz energy functional $\mathcal{L}$, more precisely $0 \in \partial\mathcal{L}(u, \phi)$, one can prove that the pair (u, ϕ) is the weak solution of the inclusion system, see section 2.4.1.

On the other hand, the pair (u, ϕ) is the critical point of $\mathcal{L}$ if and only if $\phi = \phi_u$ and u is the critical point of the single variable energy functional $\mathcal{E}_\lambda$. With the same notation as above, for every test function $v \in H^1(M)$ we have

$$\begin{aligned}\partial\mathcal{E}_\lambda(u)(v) = &\int_M \nabla_g u(x)\nabla_g v(x)\mathrm{d}v_g + \int_M \phi_u(x)u(x)v(x)\mathrm{d}v_g \\ &- \lambda\int_M \alpha(x)\xi v(x)\mathrm{d}v_g.\end{aligned} \tag{6.2}$$

Accordingly, in order to solve the problem $(\mathcal{S}M)$, it is enough to find critical points for the singe variable energy functional. However, due to the lack of compactness we have to use isometric actions. Since subsection 6.2.2 proves that $\mathcal{E}_\lambda$ is G-invariant, applying the non-smooth symmetric criticality principle 2.4.3 implies that if u_G is a critical point of $\mathcal{E}_{\lambda,G}$, then it is also a critical point of the locally Lipschitz energy functional $\mathcal{E}_\lambda$. Now, we are in the position to find critical points for the energy functional $\mathcal{E}_{\lambda,G}$, thus guaranteeing weak solutions to the inclusion system $(\mathcal{S}M)$.

6.2.6 First solution for large parameters

In what follows, we prove the existence of the first solution. Let $q \in (2, 2^*)$. According to assumptions $(\mathbf{F}_1) - (\mathbf{F}_3)$ and appropriate Sobolev embeddings, for all $u \in H^1_G(M)$ we have

$$\begin{aligned} 0 \leq \mathcal{F}(u) &\leq \int_M \alpha(x) F(u(x)) \mathrm{d}v_g \leq \int_M \alpha(x) |F(u(x))| \mathrm{d}v_g \\ &\leq \|\alpha\|_{L^\infty} (\varepsilon \|u\|^2_{H^1} + l_\varepsilon (K^\pm_q)^q \|u\|^q_{H^1}), \end{aligned}$$

where $K^\pm_q$ are the continuous embedding constants.

Since

$$\mathcal{N}(u) = \frac{1}{2}\|u(x)\|^2_{H^1} + \frac{1}{2}\int_M \phi_u(x) u^2(x) \mathrm{d}v_g,$$

it follows that $\lim_{\|u\|^2_{H^1} \to 0} \frac{\mathcal{F}(u)}{\mathcal{N}(u)} = 0$ and $\lim_{\|u\|^2_{H^1} \to \infty} \frac{\mathcal{F}(u)}{\mathcal{N}(u)} = 0$. Consequently, one has a number $\lambda_1 > 0$ such that

$$0 < \lambda_1 = \inf_{\mathcal{F}(u) > 0} \frac{\mathcal{N}(u)}{\mathcal{F}(u)} < \infty;$$

if $\lambda > \lambda_1$, one can find $\omega \in H^1_G(M)$ such that $\mathcal{F}(\omega) > 0$ and $\lambda > \frac{\mathcal{N}(\omega)}{\mathcal{F}(\omega)} > \lambda_1$. It means, that

$$C^1_\lambda = \inf_{u \in H^1_G(M)} \mathcal{E}_{\lambda,G}(u) \leq \mathcal{N}(\omega) - \lambda \mathcal{F}(\omega) < 0.$$

We have shown that the energy functional $\mathcal{E}_{\lambda,G}$ is bounded from below, and satisfies the Palais-Smale condition, thus C^1_λ is a critical value. In particular, we can garantee a critical point $u^1_\lambda \in H^1_G(M)$ for the energy functional $\mathcal{E}_{\lambda,G}$ such that $\mathcal{E}_{\lambda,G}(u^1_\lambda) < 0 = \mathcal{E}_{\lambda,G}(0)$. According to section 6.2.5, u^1_λ is a non-trivial weak solution to the problem $(\mathcal{S}M)$.

6.2.7 Second solution for large parameters

This subsection is devoted to present our second non-trivial weak solution. Let $q \in (2, 2^*)$ and $\lambda > \lambda_1$. One has that

$$\begin{aligned}\mathcal{E}_{\lambda,G}(u) &= \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2}\int_M \phi_u(x)u^2(x)\mathrm{d}v_g - \lambda\mathcal{F}(u)\\ &\geq \|u\|_{H^1}^2\left(\frac{1}{2} - \lambda\varepsilon\|\alpha\|_{L^\infty}\right) - \lambda\|\alpha\|_{L^\infty} l_\varepsilon(K_q^{\pm})^q\|u\|_{H^1}^q.\end{aligned}$$

Fixing a small $\varepsilon > 0$ (e.g. $\varepsilon < \frac{1}{2\lambda\|\alpha\|_{L^\infty}}$), one has $\theta > 0$, e.g. $\theta = \left(\frac{1-2\lambda\varepsilon\|\alpha\|_{L^\infty}}{3\lambda\|\alpha\|_{L^\infty} l_\varepsilon(K_q^{\pm})^q}\right)^{\frac{1}{q-2}}$ such that

$$\inf_{\|u\|_{H^1}=\theta} \mathcal{E}_{\lambda,G}(u) > \mathcal{E}_{\lambda,G}(0) = 0 > \mathcal{E}_{\lambda,G}(\omega).$$

Consequently, the energy functional $\mathcal{E}_{\lambda,G}$ satisfies the mountain pass geometry. Section 6.2.4 proves that $\mathcal{E}_{\lambda,G}$ fulfills the non-smooth Palais-Smale condition, thus applying the non-smooth mountain pass theorem we can guarantee another critical point $u_\lambda^2 \in H_G^1(M)$, i.e.,

$$0 \in \partial\mathcal{E}_{\lambda,G}(u_\lambda^2),$$

$$\mathcal{E}_{\lambda,G}(u_\lambda^2) = C_\lambda^2 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{E}_{\lambda,G}(\gamma(t)) > 0,$$

where Γ is the set of continuous functions given by

$$\Gamma = \left\{\gamma \in C([0,1]; H_G^1(M)) : \gamma(0) = 0, \gamma(1) = \omega\right\}.$$

Based on the fact that $C_\lambda^1 < 0 < C_\lambda^2$, section 6.2.5 implies that we have a second nontrivial weak solution u_λ^2 for the inclusion system $(\mathcal{S}M)$.

6.2.8 Non-existence of solutions for small parameters

We are going to show that the inclusion system $(\mathcal{S}M)$ has just the trivial solution whenever $0 < \lambda < \lambda_0$, where λ_0 will be defined later. Let assume that $(u, \phi_u) \in (H^1(M) \times H^1(M))$ is the weak, non-trivial solution of the problem. Recalling (6.2) with $v = u$ gives for all $u \in H^1(M)$ that

$$\|u\|_{H^1(M)}^2 + \int_M \phi_u(x)u^2(x)\mathrm{d}v_g = \lambda\int_M \alpha(x)\xi u(x)\mathrm{d}v_g,$$

where $\xi \in \partial F(u)$ and $x \in (M, g)$. Accordingly, by combining assumptions $(\mathbf{F}_1) - (\mathbf{F}_3)$ with the fact that ∂F is upper semicontinuous, one has a constant $K_1 > 0$ such that

$|\xi| \leq K_1 s$ for all $\xi \in \partial F(s)$, $s > 0$. Consequently, we obtain that

$$\|u\|^2_{H^1(M)} + \int_M \phi_u(x)u^2(x)\mathrm{d}v_g \leq \lambda\|\alpha\|_{L^\infty}K_1\|u\|^2_{H^1}.$$

If $0 \leq \lambda < \frac{1}{\|\alpha\|_\infty K_1} =: \lambda_0$, then the inequality has just the trivial $u = 0$ solution. Thus, the Maxwell equation also has just the trivial solution $\phi_u = 0$. Indeed, we can just garantee the trivial solution pair $(u = 0, \phi_u = 0)$ for the inclusion system $(\mathcal{S}M)$, which clearly proves our claim.

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