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Non-smooth elliptic problems on smooth manifolds

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Chapter 1

Introduction

The study of partial differential equations had already appeared within the analysis of physical models in the works of Euler, Lagrange and Laplace by the 18th century. Motivated by mathematical and physical problems, PDEs became an essential research area, both as standalone mathematical discipline as well as modeling various problems in physics, providing a bridge between pure mathematics and applications.

Arising in the context of several natural phenomena, PDEs have some well known, famous applications, like wave, Schrödinger, Maxwell, diffusion, Monge-Ampère and Navier-Stokes equations, respectively. The Laplace equation modeling the stationary state of the heat equation is the most simple variant of the elliptic class of PDEs besides the Poisson equation. The elliptic class of problems being generalization of the Laplace equation, is suitable to describe equilibrium states, or problems which are independent from the time. Mechanical or physical applications often induce not only continuous, but also discontinuous functions, where the idea is to ”fill the gaps” of the discontinuities with a set-valued generalized gradient of a locally Lipschitz function, e.g. von Kármán laminated plates problem, where the external force acts on adhesively connected laminated plates, analysed by Bocea, Panagiotopoulos and Rădulescu [6]. In this way, the appearance of non-smooth problems (thus, set-valued mappings) induces differential inclusions rather than differential equations.

Elliptic PDEs are usually studied on Sobolev spaces combined with powerful variational methods. Analyzing some fine properties of the energy functional associated to the studied problem, and exploiting variational methods like minimax or minimization principles, we may find critical points and prove in this way existence, uniqueness and multiplicity results. In case of discontinuous functions, non-smooth variational methods should be applied.

The primary objective of the thesis is to present recent research results in the study of elliptic differential inclusions. Applying recent geometrical researches, we show how to apply variational methods not only on Euclidean spaces but also on curved cases.

The thesis is based on the following papers:

(i) A. Kristály, I.I. Mezei and K. Szilák. *Differential inclusions involving oscillatory term.* Nonlinear Analysis, 197 (2020), 111834. [**D1** publication]

(ii) K. Szilák. *A non-smooth Neumann problem on compact Riemannian manifolds.* SACI 2021 IEEE 15th International Symposium on Applied Computational Intelligence and Informatics.

(iii) A. Kristály, I.I. Mezei and K. Szilák. *Elliptic differential inclusions on non-compact Riemannian manifolds.* Nonlinear Analysis-Real World Applications, 69 (2023), 103740. [**D1** publication]

(iv) K. Szilák. *Schrödinger-Maxwell differential inclusion system.* SACI 2023 IEEE 17th International Symposium on Applied Computational Intelligence and Informatics.

(v) Á. Mester, K. Szilák, A Dirichlet inclusion problem on Finsler manifolds, CINTI 2023, IEEE 23rd International Symposium on Computational Intelligence and Informatics, November 20-22, 2023, Budapest, Hungary.

In the sequel a brief overview follows about the chapters. In Chapter 2, motivated by mechanical problems – where the external forces are non-smooth – we study an elliptic inclusion problem with a non-smooth oscillatory and a non-smooth, generic, p-order perturbation function in two settings. First, we consider the case when the oscillatory term oscillates near to the origin and the perturbation is of order $p > 0$ at origin. Applying various non-smooth variational methods, we provide a quite complete picture about the number of distinct, non-trivial weak solutions for the studied problem, depending on parameters p, λ and k, and we also prove a novel competition phenomena. As a counterpart, we also prove similar results whenever the nonlinear term oscillates at infinity and the perturbation is of order $p > 0$ at infinity. This chapter is based on the paper by Kristály, Mezei and Szilák [11].

In Chapter 3, considering a non-smooth elliptic problem on Riemannian manifolds, we discuss a differential inclusion, as a new application of a recent non-smooth Ricceri-type result. We prove that the studied inclusion problem has at least three distinct weak solutions whose norms are controlled whenever a suitable perturbation occurs. This chapter is based on Szilák [23].

Chapter 4 is devoted to focus onto a broad class of curved spaces. More precisely, we consider both Cartan-Hadamard manifolds and non-compact Riemannian manifolds with non-negative Ricci curvature. Within these geometric settings, we study an elliptic inclusion problem involving a singular term and a non-smooth nonlinearity, by proving various non-existence and existence results. In particular, four non-trivial G-invariant weak solutions are established in the above two settings (where G is a certain subgroup of isometries of the Riemannian manifold). In the first case, the nonlinear term is sub-quadratic, meanwhile in the second case it is super-quadratic at infinity. It turns out that the usual variational methods cannot be applied due to the lack of compactness, which will be recovered by isometric actions, combined with the principle of symmetric criticality. This chapter is based on the paper by Kristály, Mezei and Szilák [12].

In Chapter 5, motivated by physical problems, we consider a Schrödinger-Maxwell inclusion system involving a non-linear term, which is superlinear at the origin and sub-linear at infinity. Similarly to Chapter 4, we again focus on Cartan-Hadamard manifolds and non-compact Riemannian manifolds with non-negative Ricci curvature, respectively. Introducing a ”single variable” energy functional, we prove a non-existence result whenever the parameter λ is small enough, and by compensating the lack of compactness with isometric actions, we establish two non-trivial weak solutions for the inclusion system whenever the parameter λ is large enough. This chapter is based on Szilák [24].

Part I

Differential inclusions - compact case

Chapter 2

Differential inclusions involving oscillatory terms

PDEs with perturbations that play central roles in physical and mechanical problems, have been subject of several investigations. Let consider the following elliptic PDE with perturbation

$$\begin{cases} -\Delta u(x) = f(u(x)) + \lambda g(u(x)), & x \in \Omega; \\ u \geq 0, & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P_λ}$$

where Δ is the usual Laplace operator, $\Omega \subset \mathbb{R}^n$ is a bounded open domain $(n \geq 2)$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function verifying certain growth conditions at the origin and infinity, $g : \mathbb{R} \to \mathbb{R}$ is another continuous function which is going to compete with the original function f. When both functions f and g are of *polynomial type* of sub- and super-unit degree, the existence of at least one or two nontrivial solutions of (P_λ) is guaranteed, depending on the range of $\lambda > 0$, see e.g. Ambrosetti, Brezis and Cerami [1], Autuori and Pucci [2], de Figueiredo, Gossez and Ubilla [8]. In these papers variational arguments, sub- and super-solution methods as well as fixed point arguments are employed.

Another important class of problems of the type (P_λ) is studied whenever f has a certain *oscillation* (near the origin or at infinity) and g is a *perturbation*.

Although oscillatory functions seemingly call forth the existence of infinitely many solutions, it turns out that 'too classical' oscillatory functions do not have such a feature. Indeed, when $f(s) = c \sin s$ and $g = 0$, with $c > 0$ small enough, a simple use of the Poincaré inequality implies that problem (P_λ) has only the zero solution. However, when

f *strongly* oscillates, problem (P_λ) with 0 perturbation has indeed infinitely many different solutions; see e.g. Omari and Zanolin [19], Saint Raymond [22]. A novel competition phenomena for the case $g(s) = s^p$ $(s > 0)$ has been described for (P_λ) by Kristály and Moroşanu [13].

In mechanical applications, in turn, the perturbation may manifest in a *discontinuous* manner as a non-regular external force, see e.g. the gluing force in von Kármán laminated plates, cf. Bocea, Panagiotopoulos and Rădulescu [6], Motreanu and Panagiotopoulos [18] and Panagiotopoulos [20]. We consider the problem (P_λ) formulated into a more general form

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)), & x \in \Omega; \\ u \geq 0, & x \in \Omega; \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{$\mathcal{D}_\lambda$}$$

where F and G are both non-smooth, locally Lipschitz functions having various growths, while ∂F and ∂G stand for the generalized gradients of F and G, respectively.

Extending the main results of Kristály and Moroşanu [13] we study the inclusion $(\mathcal{D}_\lambda)$ in two different settings, i.e., we analyze the number of distinct solutions of $(\mathcal{D}_\lambda)$ whenever ∂F oscillates near the origin/infinity and ∂G is of order $p > 0$ near the origin/infinity.

2.1 Main theorems

Let $F, G : \mathbb{R}_+ \to \mathbb{R}$ be locally Lipschitz functions and as usual, let us denote by ∂F and ∂G their generalized gradients in the sense of Clarke. Hereafter, $\mathbb{R}_+ = [0, \infty)$. Let $p > 0$, $\lambda \geq 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded open domain, and consider the elliptic differential inclusion problem

$$\begin{cases} -\Delta u(x) \in \partial F(u(x)) + \lambda \partial G(u(x)), & x \in \Omega; \\ u \geq 0, & x \in \Omega; \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{$\mathcal{D}_\lambda$}$$

The cases when ∂F oscillates near the *origin* or at *infinity* are studied in separated sections.

2.1.1 Oscillation near the origin

We assume that the beforementioned locally Lipschitz functions F and G satisfy the following conditions:

$(\mathbf{F}_0^0):\ F(0)=0;$

$(\mathbf{F}_1^0):\ -\infty < \liminf_{s\to 0^+}\frac{F(s)}{s^2};\ \limsup_{s\to 0^+}\frac{F(s)}{s^2}=+\infty;$

$(\mathbf{F}_2^0):\ l_0 := \liminf_{s\to 0^+}\frac{\max\{\xi:\xi\in\partial F(s)\}}{s}<0;$

$(\mathbf{G}_0^0):\ G(0)=0;$

$(\mathbf{G}_1^0):$ There exist $p>0$ and $\underline{c},\overline{c}\in\mathbb{R}$ such that

$$\underline{c}=\liminf_{s\to 0^+}\frac{\min\{\xi:\xi\in\partial G(s)\}}{s^p}\le\limsup_{s\to 0^+}\frac{\max\{\xi:\xi\in\partial G(s)\}}{s^p}=\overline{c}.$$

Remark 2.1.1. Hypotheses $(\mathbf{F}_1^0)$ and $(\mathbf{F}_2^0)$ imply a strong oscillatory behavior of ∂F near the origin.

In what follows, we provide a quite complete picture about the competition concerning the terms $s\mapsto\partial F(s)$ and $s\mapsto\partial G(s)$, respectively. First, we are going to show that when $p\geq 1$, then the 'leading' term is the oscillatory function ∂F; roughly speaking, one can say that the effect of $s\mapsto\partial G(s)$ is negligible in this competition. More precisely, we prove the following result.

Theorem 2.1.1. (Kristály, Mezei and Szilák [11]) (Case $p\geq 1$) *Assume that $p\geq 1$ and the locally Lipschitz functions $F,G:\mathbb{R}_+\to\mathbb{R}$ satisfy $(\mathbf{F}_0^0)-(\mathbf{F}_2^0)$ and $(\mathbf{G}_0^0)-(\mathbf{G}_1^0)$. If* (i) *either $p=1$ and $\lambda\overline{c}<-l_0$ (with $\lambda\geq 0$),* (ii) *or $p>1$ and $\lambda\geq 0$ is arbitrary,*
then the differential inclusion problem $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i\subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i\to\infty}\|u_i\|_{H_0^1}=\lim_{i\to\infty}\|u_i\|_{L^\infty}=0.$$

In the case when $p<1$, the perturbation term ∂G may compete with the oscillatory function ∂F; we have the following theorem:

Theorem 2.1.2. (Kristály, Mezei and Szilák [11]) (Case $0<p<1$) *Assume $0<p<1$ and that the locally Lipschitz functions $F,G:\mathbb{R}_+\to\mathbb{R}$ satisfy $(\mathbf{F}_0^0)-(\mathbf{F}_2^0)$ and $(\mathbf{G}_0^0)-(\mathbf{G}_1^0)$.*

Then, for every $k \in \mathbb{N}$, there exists $\lambda_k > 0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least k distinct weak solutions $\{u_{1,\lambda}, ..., u_{k,\lambda}\} \subset H_0^1(\Omega)$ whenever $\lambda \in [0, \lambda_k]$. Moreover,

$$\|u_{i,\lambda}\|_{H_0^1} < i^{-1} \ and \ \|u_{i,\lambda}\|_{L^\infty} < i^{-1} \ for \ any \ i = \overline{1,k}; \ \lambda \in [0, \lambda_k]. \tag{2.1}$$

2.1.2 Oscillation at infinity

We assume that the beforementioned locally Lipschitz functions F and G satisfy the following conditions:

$(\mathbf{F}_0^\infty)$: $F(0) = 0$;

$(\mathbf{F}_1^\infty)$: $-\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2}$; $\limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty$;

$(\mathbf{F}_2^\infty)$: $l_\infty := \liminf_{s\to\infty} \frac{\max\{\xi:\xi\in\partial F(s)\}}{s} < 0$;

$(\mathbf{G}_0^\infty)$: $G(0) = 0$;

$(\mathbf{G}_1^\infty)$: There exist $p > 0$ and $\underline{c}, \overline{c} \in \mathbb{R}$ such that

$$\underline{c} = \liminf_{s\to\infty} \frac{\min\{\xi : \xi \in \partial G(s)\}}{s^p} \le \limsup_{s\to\infty} \frac{\max\{\xi : \xi \in \partial G(s)\}}{s^p} = \overline{c}.$$

Remark 2.1.2. Hypotheses $(\mathbf{F}_1^\infty)$ and $(\mathbf{F}_2^\infty)$ imply a strong oscillatory behavior of the set-valued map ∂F at infinity.

In the sequel, we investigate the competition at infinity concerning the terms $s \mapsto \partial F(s)$ and $s \mapsto \partial G(s)$, respectively. First, we show that when $p \le 1$ then the 'leading' term is the oscillatory function F, i.e., the effect of $s \mapsto \partial G(s)$ is negligible. More precisely, we prove the following result:

Theorem 2.1.3. (Kristály, Mezei and Szilák [11]) (Case $p \le 1$) *Assume that $p \le 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(\mathbf{F}_0^\infty) - (\mathbf{F}_2^\infty)$ and $(\mathbf{G}_0^\infty) - (\mathbf{G}_1^\infty)$. If*

(i) *either $p = 1$ and $\lambda\overline{c} \le -l_0$ (with $\lambda \ge 0$),*

(ii) *or $p < 1$ and $\lambda \ge 0$ is arbitrary,*

then the differential inclusion $(\mathcal{D}_\lambda)$ admits a sequence $\{u_i\}_i \subset H_0^1(\Omega)$ of distinct weak solutions such that

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{2.2}$$

Remark 2.1.3. Let us denote by 2^* the usual critical Sobolev exponent. In addition to (2.2), we also claim that $\lim_{i\to\infty} \|u_i^\infty\|_{H_0^1} = \infty$ whenever

$$\sup_{s\in[0,\infty)} \frac{\max\{|\xi| : \xi \in \partial F(s)\}}{1+s^{2^*-1}} < \infty. \tag{2.3}$$

In the case when $p > 1$, it turns out that the perturbation term ∂G may compete with the oscillatory function ∂F; more precisely, we have the following theorem:

Theorem 2.1.4. (Kristály, Mezei and Szilák [11]) (Case $p > 1$) *Assume that $p > 1$ and the locally Lipschitz functions $F, G : \mathbb{R}_+ \to \mathbb{R}$ satisfy $(\mathbf{F}_0^\infty) - (\mathbf{F}_2^\infty)$ and $(\mathbf{G}_0^\infty) - (\mathbf{G}_1^\infty)$. Then, for every $k \in \mathbb{N}$, there exists $\lambda_k^\infty > 0$ such that the differential inclusion $(\mathcal{D}_\lambda)$ has at least k distinct weak solutions $\{u_{1,\lambda}, ..., u_{k,\lambda}\} \subset H_0^1(\Omega)$ whenever $\lambda \in [0, \lambda_k^\infty]$. Moreover,*

$$\|u_{i,\lambda}\|_{L^\infty} > i-1 \ \ for\ any\ i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty]. \tag{2.4}$$

Remark 2.1.4. If the condition (2.3) holds and $p \le 2^* - 1$ in Theorem 2.3, then we claim in addition that

$$\|u_{i,\lambda}^\infty\|_{H_0^1} > i-1 \ \text{ for any } i = \overline{1,k};\ \lambda \in [0, \lambda_k^\infty].$$

Chapter 3

A non-smooth Neumann problem on compact Riemannian manifolds

In many cases, a recent Ricceri result [21] can be easily invoked to solve partial differential equations involving C^1 functions; for a non-smooth version, see Kristály, Marzantowicz and Varga [10]. Extending their results in several aspects, the aim of this chapter is to present an application of the non-smooth Ricceri's multiplicity theorem [10] to discuss a differential inclusion problem on a compact Riemannian manifolds.

This chapter summerizes results of Szilák [23].

3.1 Main results

Let (M, g) be a connected, compact Riemannian manifold of dimension $n \geq 3$ with boundary ∂M. Introducing the notations $2^* = \frac{2n}{n-2}$ and $\overline{2}^* = \frac{2(n-1)}{n-2}$, we study the following inhomogeneous Neumann boundary differential inclusion problem

$$\begin{cases} -\Delta_g u(x) + k(x)u \in \lambda K(x)\partial F(u(x)), & x \in M; \\ \frac{\partial u}{\partial \mathbf{n}} \in \mu D(x)\partial G(u(x)), & x \in \partial M, \end{cases} \tag{$\mathcal{D}_{\lambda,\mu}$}$$

where $k, K : M \to \mathbb{R}$ and $D : \partial M \to \mathbb{R}$ are positive continuous functions, μ and $\lambda > 0$, Δ_g denotes the Laplace-Beltrami operator on (M, g), $\frac{\partial}{\partial \mathbf{n}}$ is the normal derivative with respect to the outward normal $\boldsymbol{n}$ on ∂M. In addition, F and G are locally Lipschitz functions, ∂F and ∂G denote their generalized gradients in the sense of Clarke and we assume they verify the following conditions:

$(\mathbf{F}_0)$: $F(0) = 0$ and there exists $C_1 > 0$ and $p \in [2, 2^*)$ such that

$$|\xi| \leq C_1(1 + |s|^{p-1}), \forall \xi \in \partial F(s), s \in \mathbb{R};$$

$(\mathbf{F}_1)$: $\limsup_{s \to 0} \frac{\max\{|\xi| : \xi \in \partial F(s)\}}{s} = 0;$

$(\mathbf{F}_2)$: $\limsup_{|s| \to \infty} \frac{F(s)}{s^2} \leq 0;$

$(\mathbf{F}_3)$: there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0;$

$(\mathbf{G}_0)$: there exists $C_2 > 0$ and $q \in [2, \overline{2}^*)$ such that

$$|\xi| \leq C_2(1 + |s|^{q-1}), \forall \xi \in \partial G(s), s \in \mathbb{R}.$$

We present the main result of this chapter:

Theorem 3.1.1. (Szilák [23]) *Let $F : M \to \mathbb{R}$ and $G : M \to R$ be functions that fulfill the assumptions $(\mathbf{F}_0) - (\mathbf{F}_3)$ and $(\mathbf{H}_0)$, respectively. Then there exist a number η and a non-degenerate compact interval $A \subset (0, +\infty)$ such that for every $\lambda \in A$ there exists $\mu_0 \in (0, \lambda + 1]$ so that whenever μ is small enough i.e. $\mu \in [0, \mu_0]$, the inclusion $(\mathcal{D}_{\lambda,\mu})$ has at least three solutions which are in norm less than η.*

Part II

Differential inclusions - non-compact case

Chapter 4

Elliptic differential inclusions on non-compact Riemannian manifolds

PDEs may appear not only on bounded domains of Euclidean structures; physical and mechanical phenomena quite frequently require the application of inclusion problems on the broad class of curved spaces. Considering a complete, n-dimensional, non-compact Riemannian manifold (M, g) with certain curvature restrictions ($n \geq 3$), we study the following differential inclusion problem

$$\mathcal{L}u(x) = -\Delta_g u(x) - \mu \frac{u(x)}{d_g^2(x_0, x)} + u(x) \in \lambda\alpha(x)\partial F(u(x)), \ \ x \in M. \qquad (\mathcal{D})$$

Here $\mathcal{L}$ denotes an elliptic type operator, Δ_g represents the Laplace-Beltrami operator on (M, g), $d_g : M \times M \to \mathbb{R}$ is the distance function associated with the Riemannian metric g, $x_0 \in M$ is a fixed point, $\mu, \lambda \in \mathbb{R}$ are some parameters. The function $\alpha : M \to \mathbb{R}$ is a measurable potential, $F : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function and ∂F stands for the Clarke subdifferential of F.

On one hand, variational elliptic differential inclusions as $(\mathcal{D})$ – or slightly different versions of them formulated in terms of variational-hemivariational inequalities – have been subject of several investigation in the last three decades, mostly in Euclidean spaces (both for bounded and unbounded domains), see e.g. Kristály and Varga [15], Liu, Liu and Motreanu [16], Liu, Livrea, Motreanu and Zeng [17], etc. On the other hand, various forms of $(\mathcal{D})$ have been investigated both on compact and non-compact Riemannian manifolds (mostly without the singular term), see e.g. Berchio, Ferrero and Grillo [4], Bonanno, Molica Bisci and Rădulescu [5], etc.

We consider problem $(\mathcal{D})$ under two different curvature conditions. More precisely, we assume that the Riemannian manifold (M, g) satisfies one of the following conditions:

(i) Cartan-Hadamard manifold,

(ii) The Ricci curvature is non-negative.

This chapter is devoted to focus on non-existence, existence and multiplicity results for the differential inclusion problem $(\mathcal{D})$ by assuming curvature hypothesis (i) or (ii), together with additional grows conditions on the locally Lipschitz function F (at the origin and infinity). It turns out that the variational methods cannot be used directly. Indeed, since such manifolds are not compact, it is not possible to use certain Sobolev embeddings; as we mentioned in the introduction, the lack of compactness has to be compensated with the application of isometric actions and the principle of symmetric criticality.

This chapter summerize results of Kristály, Mezei, and Szilák [12].

4.1 Main theorems

First, we discuss non-existence results under the above special curvature conditions; to do this, we assume on the potential $\alpha : M \to \mathbb{R}$ that

$(\mathbf{H}_\alpha)$: $\alpha \geq 0$ and $\alpha \in L^1(M) \cap L^\infty(M) \setminus \{0\}$,

and additionally on the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ that

$(\mathbf{H}_0)$: there exists $C_0 > 0$ such that

$$|\xi| \leq C_0|t|, \ \ \forall \xi \in \partial F(t), \ t \in \mathbb{R}.$$

The first result of the present chapter reads as follows.

Theorem 4.1.1. (Kristály, Mezei, and Szilák [12]) (Non-existence) *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and assume that the potential $\alpha : M \to \mathbb{R}$ and the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ satisfy assumptions $(\mathbf{H}_\alpha)$ and $(\mathbf{H}_0)$, respectively. Assume in addition that one of the following curvature conditions holds*:

(i) $\mathbf{K} \leq -\kappa$ *for some $\kappa \geq 0$, (M, g) is simply connected and*

(i1) either $\kappa = 0$, $\mu \leq \frac{(n-2)^2}{4}$ and $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$,

(i2) or $\kappa > 0$, $\mu \leq \frac{(n-2)^2}{4}$ *and* $(n-2)^2(|\lambda| C_0 \|\alpha\|_{L^\infty} - 1) \leq (n-1)^2 \left(\frac{(n-2)^2}{4} - \mu_+\right) \kappa$, *where* $\mu_+ = \max(\mu, 0)$;

(ii) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mu \leq \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$ *and* $|\lambda| C_0 \|\alpha\|_{L^\infty} \leq 1$.

Then the differential inclusion $(\mathcal{D})$ *has only the zero solution.*

In order to produce existence or even multiplicity of non-zero solutions to $(\mathcal{D})$, we require on the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ the following assumptions:

$(\mathbf{H})_1 : \lim\limits_{t \to 0} \dfrac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0;$

$(\mathbf{H})_2 : \lim\limits_{|t| \to \infty} \dfrac{\max\{|\xi| : \xi \in \partial F(t)\}}{t} = 0;$

$(\mathbf{H})_3 : F(0) = 0$ and there exist $t_0^- < 0 < t_0^+$ such that $F(t_0^\pm) > 0$.

Remark 4.1.1. Note that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ mean that the function $t \mapsto \max\{|\xi| : \xi \in \partial F(t)\}$ is *superlinear at the origin* and *sublinear at infinity*, respectively; in particular, by using Lebourg's mean value theorem, we observe that F is *sub-quadratic at infinity*.

Remark 4.1.2. By the upper semicontinuity of the set-valued function $t \mapsto \partial F(t)$ and conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, we can observe that the hypothesis $(\mathbf{H}_0)$ is also valid for a suitably large value of $C_0 > 0$; in particular, Theorem 4.1.1 can be applied (under the assumptions $(\mathbf{H}_\lambda)$, $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$), and for sufficiently 'small' values of $|\lambda|$ only the zero solution exists for the differential inclusion $(\mathcal{D})$.

Whenever λ is large enough, multiplicity result can be established involving additional assumptions in order to balance the lack of compactness of the Riemannian manifolds we are dealing with. The next theorem provides a multiplicity result with a sub-quadratic nonlinearity at the infinity.

Theorem 4.1.2. (Kristály, Mezei, and Szilák [12]) (Multiplicity: sub-quadratic nonlinearity at infinity) *Let* (M, g) *be an* n*-dimensional complete non-compact Riemannian manifold,* $n \geq 3$*, and* G *be a compact connected subgroup of* $\mathsf{Isom}_g(M)$ *such that* $\mathsf{Fix}_M(G) = \{x_0\}$ *for the same* $x_0 \in M$ *as in problem* $(\mathcal{D})$*. Let* $\alpha : M \to \mathbb{R}$ *be a potential satisfying* $(\mathbf{H}_\alpha)$ *which depends only on* $d_g(x_0, \cdot)$ *and the locally Lipschitz function* $F : \mathbb{R} \to \mathbb{R}$ *satisfying assumptions* $(\mathbf{H}_i)$*,* $i \in \{1, 2, 3\}$*, respectively. In addition, we assume that one of the following curvature assumptions holds*:

(i) (M, g) *is of Cartan-Hadamard-type and*

(ii) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mathsf{AVR}_{(M,g)} > 0$, $0 \leq \mu < \mathsf{AVR}_{(M,g)}^{\frac{2}{n}} \frac{(n-2)^2}{4}$ *and* G *is coercive.*

Then there exists $\lambda_0 > 0$ *such that for every* $\lambda > \lambda_0$ *the differential inclusion* $(\mathcal{D})$ *has at least four non-zero* G*-invariant solutions in* $H^1(M)$.

In the sequel, we establish a counterpart of Theorem 4.1.2 whenever F is *super-quadratic at infinity*. In order to prove more existence and multiplicity results, we introduce additional constraints on the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$:

$(\mathbf{H}_4)$: $F(0) = 0$ and there exist $\nu > 2$ and $C > 0$ such that

$$2F(t) + F^0(t; -t) \leq -C|t|^\nu, \ \ \forall t \in \mathbb{R};$$

$(\mathbf{H}_5)$: there is $q \in \left(2, 2 + \frac{4}{n}\right)$ such that $\max\{|\xi| : \xi \in \partial F(t)\} = O(|t|^{q-1})$ as $|t| \to \infty$. Here, $F^0(t; s)$ is the generalized directional derivative of F at the point $t \in \mathbb{R}$ and direction $s \in \mathbb{R}$. Note that by $(\mathbf{H}_1)$ and $(\mathbf{H}_4)$, F is super-quadratic at infinity.

Theorem 4.1.3. (Kristály, Mezei, and Szilák [12]) (Existence/Multiplicity: super-quadratic nonlinearity at infinity) *Let* (M, g) *be an* n*-dimensional complete non-compact Riemannian manifold,* $n \geq 3$*, and* G *be a compact connected subgroup of* $\mathsf{Isom}_g(M)$ *such that* $\mathsf{Fix}_M(G) = \{x_0\}$ *for the same* $x_0 \in M$ *as in problem* $(\mathcal{D})$*. Let* $\alpha \in L^\infty(M)$ *be a potential which depends only on* $d_g(x_0, \cdot)$ *and* $\mathrm{essinf}_{x\in M}\alpha(x) = \alpha_0 > 0$*, while the locally Lipschitz function* $F : \mathbb{R} \to \mathbb{R}$ *satisfies the assumptions* $(\mathbf{H}_i)$*,* $i \in \{1, 4, 5\}$*, respectively. If one of the curvature assumptions* (i) *or* (ii) *holds from Theorem 4.1.2, then for every* $\lambda > 0$ *the differential inclusion* $(\mathcal{D})$ *has at least a non-zero* G*-invariant solution in* $H^1(M)$*. In addition, if* F *is an even function,* $(\mathcal{D})$ *has infinitely many distinct* G*-invariant solutions in* $H^1(M)$.

Chapter 5

Schrödinger-Maxwell differential inclusion system

Electrostatic variations of Schrödinger-Maxwell systems have been subject of several investigations, see Kristály, Repovš [14], Azzollini and Pimenta [3], describing a charged quantum-mechanical particle interacting with the electromagnetic field:

$$\begin{cases} -\Delta u(x) + eu(x)\phi(x) = f(u(x)), & x \in \mathbb{R}^3; \\ -\Delta\phi(x) = eu^2(x), & x \in \mathbb{R}^3, \end{cases}$$

where Δ denotes the Laplace operator, $\phi : \mathbb{R}^3 \to \mathbb{R}$ is the electric potential, e is the electron charge constant, while the function $u : \mathbb{R}^3 \to \mathbb{R}$ is the field associated to the particle. Recent researches focus to curved spaces, see e.g. Farkas and Kristály ([9]), where existence results for Schrödinger-Maxwell systems are provided on non-compact Hadamard manifolds, involving sublinear or oscillatory terms at infinity.

Considering a broad class of non-compact Riemannian manifolds, we study a non-smooth Schrödinger-Maxwell inclusion system, equipped with a nonlinear term on the non-compact Riemannian manifold (M, g), namely

$$\begin{cases} -\Delta_g u(x) + u(x) + u(x)\phi(x) \in \lambda\alpha(x)\partial F(u(x)), & x \in M; \\ -\Delta_g\phi(x) + \phi(x) = 4\pi u^2(x), & x \in M, \end{cases} \qquad (\mathcal{S}M)$$

where Δ_g denotes the Laplace-Beltrami operator on (M, g), $\lambda > 0$ is a parameter, and the unknown terms $u, \phi : M \to \mathbb{R}$. In the sequel, $\alpha : M \to \mathbb{R}$ is a potential, and ∂F stands for the generalized gradient of the locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ in the sense of Clarke (see [7]), satisfying the following conditions:

$(\mathbf{F}_1)$: $\lim_{t\to 0+} \frac{\max\{|\theta|:\theta\in\partial F(t)\}}{t} = 0$;

$(\mathbf{F}_2)$: $\lim_{t\to \infty} \frac{\max\{|\theta|:\theta\in\partial F(t)\}}{t} = 0$;

$(\mathbf{F}_3)$: $F(0) = 0$;

$(\mathbf{C}_\alpha)$: The function $\alpha : M \to \mathbb{R}$ is radially symmetric with respect to some $x_0 \in M$ and $\inf_M \alpha > 0$.

The main objective is to prove non-existence and existence results depending on the parameter λ for the inclusion system $(\mathcal{S}M)$, equipped with non-linear term, not only on Hadamard manifolds, but also on Riemann manifolds with non-negative Ricci curvature. It turns out, similarly to the chapter 4, that the lack of compactness has to be compensated by applying isometric actions and the principle of symmetric criticality in order to use variational methods. Similarly to Farkas and Kristály [9], due to the coupled system we have to introduce a "single variable" energy functional. Our aim is to extend the result of Farkas and Kristály [9] to non-smooth functions and examine the problem on two different classes of curved spaces.

This chapter is based on results proved in Szilák [24].

5.1 Main results

Our main theorem can be stated as follows:

Theorem 5.1.1. (Szilák [24]) *Let (M, g) be an n-dimensional complete non-compact Riemannian manifold, $n \geq 3$, and G be a compact connected subgroup of $\mathsf{Isom}_g(M)$, $x_0 \in M$ is fixed, and $\mathrm{Fix}_M(G) = \{x_0\}$. Let assume that the potential α and the locally Lipschitz function F satisfy hypotheses $(\mathbf{C}_\alpha)$, $(\mathbf{F}_1)$-$(\mathbf{F}_3)$, respectively. Moreover, let assume that one of the following curvature assumptions holds:*

(a) *(M, g) is Cartan-Hadamard type,*

(b) $\mathsf{Ric}_{(M,g)} \geq 0$, $\mathsf{AVR}_{(M,g)} > 0$ *and G is coercive.*

Then there exist $\lambda_1 > \lambda_0 > 0$ such that the differential inclusion system $(\mathcal{S}M)$ has only the trivial solution for $0 < \lambda < \lambda_0$, and has two different non-trivial solutions whenever $\lambda > \lambda_1$.

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